Artificial Intelligence 322 (2023) 103947

Contents lists available at ScienceDirect

Artificial Intelligence

journal homepage: www.elsevier.com/locate/artint

# On the progression of belief ☆

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ARTICLE INFO

*Article history:*
Received 13 April 2022
Received in revised form 17 April 2023
Accepted 21 May 2023
Available online 25 May 2023

*Keywords:*
Reasoning about action
Reasoning about belief
Progression of belief

ABSTRACT

Based on weighted possible-world semantics, Belle and Lakemeyer recently proposed the logic DS, a probabilistic extension of a modal variant of the situation calculus with a model of belief. The logic has many desirable properties like full introspection and it is able to precisely capture the beliefs of a probabilistic knowledge base in terms of the notion of only-believing. While the proposal is intuitively appealing, it is unclear how to do planning with such logic. The reason behind this is that the logic lacks projection reasoning mechanisms and projection lies at the heart of planning. Projection reasoning, in general, is to decide what holds after actions. Two main solutions to projection exist: regression and progression. Roughly, regression reduces a query about the future to a query about the initial state while progression, on the other hand, changes the initial state according to the effects of actions and then checks whether the formula holds in the updated state. In this paper, we study projection by progression in the logic DS. It is known that the progression of a categorical knowledge base wrt a noise-free action corresponds to what is only-known after that action. We show how to progress a type of probabilistic knowledge base wrt noisy actions by the notion of only-believing after actions. Our notion of only-believing is closely related to Lin and Reiter's notion of progression.



## 1. Introduction

Rich representation of knowledge and actions has been a goal that many AI researchers pursue. Among all proposals, the situation calculus [2] is perhaps the most widely-studied, where actions are treated as logical terms and agent's knowledge is represented by logical formulas. The language has been extended to incorporate many features like time, concurrency, procedures, etc. Later, combining it with probability, Bacchus, Halpern and Levesque (BHL) provided a rich account of dealing with degrees of belief and noisy sensing [3]. The main advantage of a logical account like BHL is that it allows partial or incomplete specifications of beliefs depending on what information is actually available in a particular domain.

Alternatively, Belle and Lakemeyer (BL) proposed a formulation [4] of BHL's ideas based on a modal variant of the situation calculus [5], extending earlier work on static probabilistic beliefs [6]. Unlike the axiomatic BHL, BL's logic $\mathcal{DS}$ is based on possible-world semantics with distributions over possible-worlds. More concretely, a distribution is just an assignment of non-negative weights to the possible worlds. An epistemic state is then defined as a set of such distributions and a sentence $\phi$ is believed with degree $r$ if and only if the normalized sum of the weights of worlds that satisfies $\phi$

☆ This paper is an invited revision of a paper which first appeared at the 2021 International Conference on Principles of Knowledge Representation and Reasoning (KR-21).

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https://doi.org/10.1016/j.artint.2023.103947

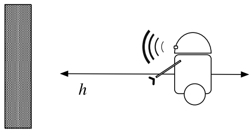


**Fig. 1.** Robot moving toward a wall.

equals $r$ in all distributions of the epistemic state. Later, beliefs after a sequence of actions are defined by the notions of *action likelihood* and *observational-indistinguishability*, which captures the idea that the agent might not be able to distinguish between certain actions.

The logic has many interesting properties such as full introspection of beliefs. Besides, it is possible to express all the agent's beliefs of a probabilistic knowledge base (KB) by appealing to a notion of *only-believing*. Nevertheless, the problem of how to plan with such logic is still open. The reason behind this is the lack of projection reasoning mechanisms and projection lies at the heart of planning. Projection reasoning, in general, is to decide what holds after actions. There are two main solutions to the projection problem: regression and progression. Roughly, regression reduces a query about the future to a query about the initial state while progression, on the other hand, changes the initial state according to the effects of actions and then checks whether the formula holds in the updated state. Compared with regression, progression is more challenging as it is proved in [7] that progression in general requires second-order logic.

Progression has been developed since then, mainly by appealing to the notion of *forgetting*. Later, Lakemeyer and Levesque showed that the progression of a categorical knowledge base specified by only-knowing wrt to a noise-free action amounts to what is only known by the agent after that action [8]. In the setting of quantitative beliefs and noisy actions, the progression would correspond to what is only believed after actions. However, the current semantics of the only-believing $\boldsymbol{O}$ in $\mathcal{DS}$ is problematic in reflecting this correctly. To see a concrete example, consider a robot moving toward a wall as in Fig. 1. Supposing that a fluent $h$ indicates the robot's distance from the wall and the robot is equipped with an accurate sonar (specified by the action model $\Sigma$), in Lakemeyer and Levesque's work, the following holds:

$$\models \boldsymbol{O}((h=1 \vee h=2) \wedge \Sigma) \supset [sonar(2)]\boldsymbol{O}(h=2 \wedge \Sigma).$$

In English, only-knowing the distance is 1 or 2 and the action model of the sonar entails that after the sonar reads 2 the agent only knows its distance is 2 (and the action model). Likewise, in a stochastic setting, one would expect that:

$$\models \boldsymbol{O}(h=1\colon 0.5,\ h=2\colon 0.5,\ \Sigma\colon 1) \supset [sonar(2)]\boldsymbol{O}(h=2 \wedge \Sigma\colon 1).$$

Namely, only-believing $h$ is among $\{1,2\}$ with equal degree, and the action model with degree one entails that after the sonar reads 2, the agent only believes $h=2$ (and the action model) with degree one. This does not follow in the logic $\mathcal{DS}$. Because the semantics of only-believing in $\mathcal{DS}$ seems to only work for the initial state and it's unclear what an epistemic state that satisfies $[sonar(2)]\boldsymbol{O}(h=2 \wedge \Sigma\colon 1)$ would look like.

Another issue is that $\mathcal{DS}$ lacks the expressiveness to specify belief distributions. Sentences like $\boldsymbol{O}((\forall u.\boldsymbol{B}(h=u\colon \mathbf{U}_{\{1,2\}}(u)) \wedge \Sigma)\colon 1)$ are unsatisfiable, even if they are intuitively reasonable (here $\mathbf{U}_{\{1,2\}}(u)$ refers to the discrete uniform distribution with points among $\{1,2\}$).

In this paper, we will address the above issues of the logic $\mathcal{DS}$ by modifying both its language and semantics, which results in a new logic $\mathcal{DS}_p$. More concretely, by special treatment of rigid terms, we are able to express distributions, for example, the above uniform distribution and geometric distributions (with expectation 2) $\forall u.\boldsymbol{B}(h=u\colon \mathcal{G}(\frac{1}{2},u))$. Besides, only-believing an arbitrary formula is satisfiable including the formula in the robot examples. By virtue of our notion of progressed distribution, we are able to fully reconstruct the results of Lakemeyer and Levesque in the new logic. It turns out that the notion of only-believing has a close relationship with the topological structure of distributions entertained by the epistemic state. For a fragment of the logic, we show that classical progression is first-order definable. Lastly, we provide our solution for the progression of belief in terms of only-believing after actions.

The paper is an extension of our conference paper [1]. Specifically, we 1) provide detailed proofs for all theorems and lemmas, etc; 2) include an expanded discussion on related works and future works; 3) add a new section to investigate thoroughly the connection between the notion of only-believing and the topological structure of distributions entertained by the epistemic state.

The rest of the paper is organized as follows. In section 2, we introduce the syntax and semantics of logic $\mathcal{DS}_p$. The semantics of progression is presented in section 3, where we address our solution of progression wrt noisy sensing and stochastic actions. In section 4 we discuss related work with a detailed discussion on the relationship between the notion of only-believing and the topological structure of distributions entertained by the epistemic state in section 5. Finally, we conclude the paper in section 6.

## 2. The logic $\mathcal{DS}_p$

Before introducing the logic $\mathcal{DS}_p$, it is instructive to review the logic $\mathcal{DS}$ informally to better understand its drawbacks and how $\mathcal{DS}_p$ overcomes them. We defer a formal review of $\mathcal{DS}$ and a comparison between $\mathcal{DS}$ and $\mathcal{DS}_p$ to Section 2.3.

### 2.1. The logic $\mathcal{DS}$ informally

The logic $\mathcal{DS}$ is a first-order modal logic with modalities of actions and beliefs. More concretely, besides usual first-order formulas, four modalities are used: $[t_a]\alpha, \Box\alpha, \boldsymbol{B}(\alpha : r)$, and $\boldsymbol{O}(\alpha_1 : r_1, \ldots, \alpha_k : r_k)$, which respectively can be read as "$\alpha$ holds after action $t_a$", "$\alpha$ holds after any action", "$\alpha$ is believed with degree $r$", and "all that is believed are $\alpha_i$ with degree $r$". Furthermore, the semantics is given in terms of *possible worlds* and a world $w$ determines the state of affairs. To define truth for beliefs, a set of distributions[1] $d$ which assign non-negative weights to worlds is used as the *epistemic state* $e$. A sentence $\alpha$ is believed with a constant degree $r$ in an epistemic state $e$, i.e. $e \models \boldsymbol{B}(\alpha : r)$, if and only if the normalized sum of the weights of worlds that satisfies $\alpha$ equals $r$ in all distributions of the epistemic state, that is, $\textsc{Norm}(d, \mathcal{W}_\alpha, \mathcal{W}_{\text{TRUE}}, r)$ for all $d \in e$, here $\mathcal{W}_\alpha$ is the set of worlds which satisfy $\alpha$, and Norm normalizes the sum of weights of worlds in $\mathcal{W}_\alpha$ against $\mathcal{W}_{\text{TRUE}}$. The semantics of $\boldsymbol{O}$ is given likewise: $e \models \boldsymbol{O}(\alpha : r)$ iff $e$ is a maximum set of distributions where $\textsc{Norm}(d, \mathcal{W}_\alpha, \mathcal{W}_{\text{TRUE}}, r)$ holds. Beliefs after actions are evaluated against pairs of worlds and actions. The ideas are as follows:

1. in the stochastic domain, stochastic actions are treated as a set of deterministic actions that are mutually alternative;
2. given a formula $\phi$, there are multiple pairs of worlds and actions that might result in $\phi$, therefore the agent considers all of them to be possible;
3. the agent believes $\phi$ with degree $r$ after stochastic action $t_a$ if and only if the normalized sum of the product of the weights of worlds and the likelihood of actions it considers to be possible equals $r$ in all distributions in its epistemic state.

We comment that the semantics of $\mathcal{DS}$ has the following issues:

- the degree of belief can only be constants which excludes the possibility to express belief distributions as aforementioned;
- it lacks an account of progression; notice that beliefs after actions are evaluated against the initial epistemic state while in typical accounts of progression, beliefs after actions are evaluated against the epistemic state that evolved according to actions, for example, by means of *forgetting* the past [7].

In our proposed logic $\mathcal{DS}_p$, we solve these issues by the following: 1) we propose a special treatment on rigid terms which then allows the degree of beliefs to be arbitrary such terms (here, rigid means the denotation is fixed); 2) based on the notion of *progressed world* and *progressed epistemic state*, we provide a semantic account for the progression of beliefs.

### 2.2. The logic $\mathcal{DS}_p$

$\mathcal{DS}_p$ is a modal language with equality and sorts of type *object* and *action*. Implicitly, we assume that *number* is a sub-sort of object and refers to the *computable numbers* $\mathbb{C}$.[2] Before presenting the formal definitions, here are the main features:

1. *standard names*: The language includes (countably many) standard names $\mathcal{N}$ for both objects $\mathcal{N}_O$ and actions $\mathcal{N}_A$ ($\mathcal{N} = \mathcal{N}_O \cup \mathcal{N}_A$). This can be viewed as having a fixed infinite domain closure axiom with the unique name assumption, which further allows first-order (FO) quantification to be understood substitutionally. Moreover, equality can also be treated in a simpler way: every ground term will have a coreferring standard name, and two terms are equal if their coreferring standard names are identical.
2. *rigid and fluent functions*: The language contains both fluent and rigid function symbols. For simplicity, all action functions are rigid and we do not include predicate symbols in the language. Fluents vary as the result of actions, yet the meaning of rigid functions is fixed.
3. *belief and truth*: The language includes modal operators $\boldsymbol{B}$ and $\boldsymbol{O}$ for degrees of belief and only-believing respectively. Such operators allow us to distinguish between sentences that are true and sentences that are believed to be true with positive degrees.
4. *observational-indistinguishability*: Finally, in uncertain domains, stochastic actions are usually treated as a set of deterministic actions that are mutually alternative and observationally-indistinguishable to the agent (as indicated by a special function $oi$). When the agent selects an action, it does not know which one is actually executed among those that are observationally indistinguishable.

[1] The idea to incorporate a set of distributions instead of a single distribution in the epistemic state derives from the philosophical stance that *de re* knowledge about degrees of belief should not be valid [9]. Namely, if epistemic states only contain single distributions, formulas such as $\exists x.\boldsymbol{K}(\boldsymbol{B}(\phi : x))$ are valid, which is counter-intuitive.

[2] We use the computable numbers as they are still enumerable and allow representing distributions mentioning real numbers such as Euler's number $e$ [10].

#### 2.2.1. The language

**Definition 1.** The symbols of $\mathcal{DS}_p$ are taken from the following vocabulary:

- first-order variables: $u, v, x, y \dots a, a' \dots$;
- second-order (SO) function variables: $F, F' \dots$;
- rigid function symbols of every arity, such as *sonar(x)*, including arithmetical functions like $+, \times, etc.$;
- fluent function symbols of every arity, such as *distanceTo(x)*, *heightOf(y)*, including a special unary symbol $l$ and a special binary symbol $oi$. Roughly, $l$ returns the likelihood of an action and $oi$ describes the observational-indistinguishability (alternative choices) among actions[3];
- connectives and other symbols: $=, \wedge, \neg, \forall, \boldsymbol{B}, \boldsymbol{O}, [\cdot], \Box$, round and square parentheses, period, colon, comma. $\boldsymbol{B}$ and $\boldsymbol{O}$ are called epistemic operators.

We remark that the logic is second-order as Lin and Reiter [7] showed that progression is, in general, only second-order definable.

**Definition 2.** The **terms** of the language are the least set of expressions such that:

- every standard name and FO variable is a term;
- if $t_1, \dots t_k$ are terms, $f$ a $k$-ary function symbol, then $f(t_1, \dots t_k)$ is a term;
- if $t_1, \dots t_k$ are terms, $F$ a $k$-ary SO variable, then $F(t_1, \dots t_k)$ is a term.

A term is said to be rigid, if and only if it does not contain fluents. *Ground terms* are terms without variables while *SO ground terms* are terms without FO variables. *Primitive terms* are terms of the form $f(n_1, \dots n_k)$, where $f$ is a function symbol and $n_i$ are object standard names. *SO primitive terms* are defined likewise by replacing $f$ with $F$, a second-order variable. We denote the sets of primitive terms of sort object and action as $\mathcal{P}_O$ and $\mathcal{P}_A$, respectively, and the set of all SO primitive terms as $\mathcal{P}_{SO}$. While object standard names are syntactically like constants, we require that action standard names are all the primitive action terms, i.e. $\mathcal{N}_A = \mathcal{P}_A$. For example, the action *sonar*(5), where a sonar returns the number 5, is considered a standard action name. Furthermore $\mathcal{Z}$ refers to the set of all finite sequences of action standard names, including the empty sequence $\langle\rangle$. We reserve standard names $\top, \bot$ in $\mathcal{N}_O$ for truth values (to simulate predicates).

**Definition 3.** The **well-formed formulas** of the language are the least set of expressions such that:

- If $t_1, t_2$ are terms, then $t_1 = t_2$ is a formula;
- If $t_a$ is a term of sort action and $\alpha$ a formula, then $[t_a]\alpha$ is a formula;
- If $\alpha$ and $\beta$ are formulas, $v$ a FO variable, $F$ a SO variable, $r, r_i$ rigid terms of sort number, then $\alpha \wedge \beta$, $\neg\alpha$, $\forall v.\alpha$, $\forall F.\alpha$, $\Box\alpha$, $\boldsymbol{B}(\alpha : r)$, and $\boldsymbol{O}(\alpha_1 : r_1, \dots, \alpha_k : r_k)$ are also formulas.

$[t_a]\alpha$ should be read as "$\alpha$ holds after action $t_a$" and $\Box\alpha$ as "$\alpha$ holds after any sequence of actions." The epistemic expression $\boldsymbol{B}(\alpha : r)$ should be read as "$\alpha$ is believed with a degree $r$". $\boldsymbol{K}\alpha$ means "$\alpha$ is known" and is an abbreviation for $\boldsymbol{B}(\alpha : 1)$. $\boldsymbol{O}(\alpha_1 : r_1, \dots \alpha_k : r_k)$ may be read as "all that is believed are $\alpha_i$ with degree $r_i$". Similarly, $\boldsymbol{O}\alpha$ means "$\alpha$ is only known" and is an abbreviation for $\boldsymbol{O}(\alpha : 1)$. For action sequence $z = a_1 \cdots a_k$, we write $[z]\alpha$ to mean $[a_1] \cdots [a_k]\alpha$. $\alpha^x_t$ is the formula obtained by substituting all free occurrences of $x$ in $\alpha$ by $t$. As usual, we treat $\alpha \vee \beta$, $\alpha \supset \beta$, $\alpha \equiv \beta$, and $\exists v.\alpha$ as abbreviations.

A *sentence* is a formula without free variables. A *semi-sentence* is a formula without free FO variables. We use TRUE as an abbreviation for $\forall x(x = x)$, and FALSE for its negation. A formula with no $\Box$ is called *bounded*. A formula with no $\Box$ or $[t_a]$ is called *static*. A formula with no $\boldsymbol{B}$ or $\boldsymbol{O}$ is called *objective*. A formula with no fluent, $\Box$ or $[t_a]$ outside $\boldsymbol{B}$ or $\boldsymbol{O}$ is called *subjective*. A formula with no $\boldsymbol{B}$, $\boldsymbol{O}$, $\Box$, $[t_a]$, $l$, $oi$ is called a *fluent formula*. A fluent formula without fluent functions is called a *rigid formula*.

#### 2.2.2. The semantics

The semantics is given in terms of *possible worlds*, which define what is true initially and after any sequence of actions. Compared to non-probabilistic accounts with deterministic actions [5], a number of challenges need to be addressed, including how to specify probabilities over *uncountably* many possible worlds, how to deal with multiple probability distributions entertained by the agent, and how to deal with probabilistic action effects which may be *unobservable* to the agent.

A **world** $w$ is a mapping from the primitive terms ($\mathcal{P}_O \cup \mathcal{P}_A$) and action sequences $\mathcal{Z}$ to standard names $\mathcal{N}$ of the right sort, satisfying:

[3] We do not include the usual *poss* function (action precondition) for simplicity.

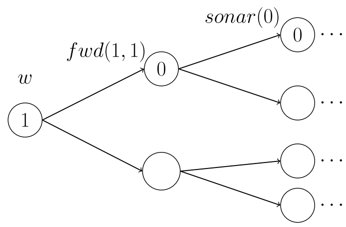


**Fig. 2.** The denotation of $h$ under a possible world $w$ for the robot example in Fig. 1.

1. **Rigidity:** If $t$ is a rigid primitive term, then for all $(w, z), (w', z')$, $w[t, z] = w'[t, z']$;
2. **Arithmetical Correctness:** Any arithmetical expression is rigid and has a value in the usual sense. For example, $w[1 + 1, z] = 2$ for all $w, z$.

Let $\mathcal{W}$ be the set of all such worlds. FO free variables are handled substitutionally by using standard names. To interpret free SO variables, we need variable maps. A variable map $\lambda$ is a mapping from $\mathcal{P}_{SO}$ to $\mathcal{N}_O$. We write $\lambda \sim_F \lambda'$ to mean $\lambda$ and $\lambda'$ agree except perhaps on SO primitives involving $F$. We now define the co-referring standard names for SO ground terms (i.e. the denotation of terms). Given SO ground terms $t, t_i, \ldots$, a world $w$, and action sequence $z$, a variable map $\lambda$, we define $|t|^z_{w,\lambda}$ (read: the co-referring standard name for $t$ given $w, z, \lambda$) recursively by:

1. If $t \in \mathcal{N}$, then $|t|^z_{w,\lambda} = t$;
2. $|f(t_1, \ldots, t_k)|^z_{w,\lambda} = w[f(|t_1|^z_{w,\lambda}, \ldots, |t_k|^z_{w,\lambda}), z]$;
3. $|F(t_1, \ldots, t_k)|^z_{w,\lambda} = \lambda[F(|t_1|^z_{w,\lambda}, \ldots, |t_k|^z_{w,\lambda})]$.

For a rigid SO ground term $t$, we use $|t|_\lambda$ instead of $|t|^z_{w,\lambda}$ for its denotation. If $t$ is a rigid first-order term, we write $|t|$. We will require that $l(a)$ is of sort number, and $oi(a, a')$ only takes values $\top$ or $\bot$, and $oi$ behaves like an equivalence relation (reflexive, symmetric, and transitive).

$oi(a, a')$ means $a$ and $a'$ are observationally indistinguishable actions. In the example of Fig. 1, the robot might perform a stochastic action $fwd(x, y)$, where $x$ is its intended forward distance and $y$ is the actual outcome selected by nature. $x$ is observable to the robot while $y$ is not. Then, $oi(fwd(1, 1), fwd(1, 0))$ says that nature can non-deterministically select 1 or 0 as a result for the intended value 1.

Fig. 2 provides an illustration of a possible world $w$ that assigns denotations for the fluent $h$ after actions. Conceptually, a possible world is a tree-like structure. In this example of possible world $w$, we have that $w[h, \langle\rangle] = 1$, $w[h, fwd(1, 1)] = 0$, and $w[h, fwd(1, 1) \cdot sonar(0)] = 0$.

By a **distribution** $d$ we mean a mapping from $\mathcal{W}$ to $\mathbb{R}^{\geq 0}$ (non-negative real) and an **epistemic state** $e$ is any set of distributions. The idea to incorporate a set of distributions instead of a single distribution in the epistemic state derives from the philosophical stance that *de re* knowledge about degrees of belief should not be valid [9]. Namely, if epistemic states only contain single distributions, formulas such as $\exists x.\boldsymbol{K}(\boldsymbol{B}(\phi : x))$ are valid, which is counter-intuitive. More concretely, in the robot example, supposing the robot's epistemic state only has one distribution $d$ about its distance $h$ (no matter how $d$ looks like), the weights of worlds that support $h = 2$ amounts to certain values in $d$, meaning formulas like $\exists x.\boldsymbol{K}(\boldsymbol{B}(h = 2 : x))$ are valid, which is counter-intuitive.

By a model, we mean a 4-tuple $(e, w, z, \lambda)$. In order to prepare for the semantics, we need to extend $l(a), oi(a, a')$ from actions to action sequences:

**Definition 4.** We define

1. $l^*: \mathcal{W} \times \mathcal{Z} \to \mathbb{R}^{\geq 0}$ as
   (a) $l^*(w, \langle\rangle) = 1$ , for every $w \in \mathcal{W}$;
   (b) $l^*(w, z \cdot a) = l^*(w, z) \times n$ where $w[l(a), z] = n$.
2. $z \sim_w z'$ as
   (a) $\langle\rangle \sim_w z'$ iff $z' = \langle\rangle$;
   (b) $z \cdot a \sim_w z'$ iff $z' = z^* \cdot a^*, z \sim_w z^*, w[oi(a, a^*), z] = \top$.

We require that $\sim_w$ is an equivalence relation, $0 \leq w[l(a), z] \leq 1$, and $\sum_{\{a' : a' \sim_w a\}} w[l(a'), z] = 1$, for all $w \in \mathcal{W}, z \in \mathcal{Z}$, and $a \in \mathcal{N}_A$. Intuitively, $l^*(w, z)$ is the likelihood of action sequence $z$ under the world $w$ while $z \sim_w z'$ means that the action sequences $z, z'$ are mutually alternatives and observationally-indistinguishable to the agent. For example, we might have a world $w$ where $fwd(1, 1)$ and $fwd(1, 0)$ are alternatives and have an equal likelihood. In such a world, $l^*(w, fwd(1, 0) \cdot fwd(1, 1)) = 0.25$ and $fwd(1, 0) \cdot fwd(1, 1) \sim_w fwd(1, 1) \cdot fwd(1, 1)$ (likewise, changing the second action to $fwd(1, 0)$,

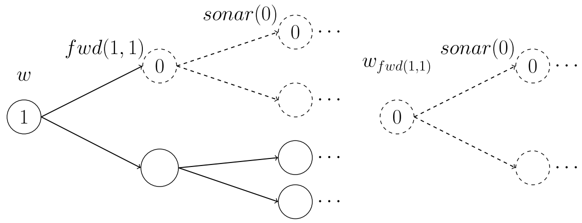


**Fig. 3.** The denotation of $h$ in the progressed world $w_{fwd(1,1)}$ (Right) of world $w$ (Left) in Fig. 2 wrt the action $fwd(1,1)$.

the same holds). Namely, the agent tries twice the forward 1 unit action, but it has no control over the outcomes of these two actions and considers all outcomes' combinations to be possible.

Since $\mathcal{W}$ is uncountable, to obtain a well-defined sum over uncountably many worlds, the following three conditions are used for evaluating beliefs [6]:

**Definition 5.** We define BND, EQ, NORM for any distribution $d$ and any set $\mathcal{V} = \{(w_1, z_1), (w_2, z_2), \ldots\}$ as follows:

1. BND$(d, \mathcal{V}, n)$ iff $\neg\exists k, (w_1, z_1), \ldots, (w_k, z_k) \in \mathcal{V}$ such that $\sum_{i=1}^{k} d(w_i) \times l^*(w_i, z_i) > n$.
2. EQ$(d, \mathcal{V}, n)$ iff BND$(d, \mathcal{V}, n)$ and there is no $n' < n$ such that BND$(d, \mathcal{V}, n')$ holds.
3. for any $\mathcal{U} \subseteq \mathcal{V}$, NORM$(d, \mathcal{U}, \mathcal{V}, n)$ iff $\exists b \neq 0$ such that EQ$(d, \mathcal{U}, b \times n)$ and EQ$(d, \mathcal{V}, b)$.

Intuitively, NORM$(d, \mathcal{U}, \mathcal{V}, n)$ says the ratio of summed weight (under distribution $d$) of worlds in $\mathcal{U}$ to worlds in $\mathcal{V}$ is $n$. EQ$(d, \mathcal{V}, n)$ and BND$(d, \mathcal{V}, n)$ express respectively that the summed weight of worlds in $\mathcal{V}$ is $n$ and is bounded by $n$. Technically, since $\mathcal{V}$ might be uncountable, the condition EQ and BND on $d$ ensure only a countable subset of $\mathcal{V}$ (the enumeration of $(w_i, z_i)$ in the definition of BND) will receive non-zero weight in $d$, hence one can construct a discrete probability space over this countable subset of $\mathcal{V}$ via $d$. Namely, $d$ is indeed a discrete probability distribution over possible worlds under the constraint of NORM (see [6] for proof).

Let $e$ be an epistemic state, $w$ a possible world, $z$ an action sequence, $\lambda$ a variable map, the truth of semi-sentences in $\mathcal{DS}_p$ is given as:

- $e, w, z, \lambda \models t_1 = t_2$ iff $|t_1|^z_{w,\lambda}$ and $|t_2|^z_{w,\lambda}$ are identical;
- $e, w, z, \lambda \models \neg\alpha$ iff $e, w, z, \lambda \not\models \alpha$;
- $e, w, z, \lambda \models \alpha \wedge \beta$ iff $e, w, z, \lambda \models \alpha$ and $e, w, z, \lambda \models \beta$;
- $e, w, z, \lambda \models \forall v.\alpha$ iff $e, w, z, \lambda \models \alpha^v_n$ for every standard name $n$ of the right sort;
- $e, w, z, \lambda \models \forall F.\alpha$ iff $e, w, z, \lambda' \models \alpha$ for all $\lambda' \sim_F \lambda$;
- $e, w, z, \lambda \models [t_a]\alpha$ iff $e, w, z \cdot n, \lambda \models \alpha$ and $n = |t_a|^z_{w,\lambda}$;
- $e, w, z, \lambda \models \Box\alpha$ iff $e, w, z \cdot z', \lambda \models \alpha$ for all $z' \in \mathcal{Z}$.

To prepare for the semantics of epistemic operators, let $\mathcal{W}^{e,z,\lambda}_\alpha = \{(w', z') \mid z' \sim_{w'} z, \text{ and } e, w', \langle\rangle, \lambda \models [z']\alpha\}$. If $z = \langle\rangle$, we ignore $z$ and write $\mathcal{W}^{e,\lambda}_\alpha$. If $\alpha$ is FO, we ignore $\lambda$ and write $\mathcal{W}^{e,z}_\alpha$. If the context is clear, we write $\mathcal{W}_\alpha$. The idea is that for a given formula $\alpha$ and action sequence $z$, the agent considers multiple pairs of possible world and action sequences, i.e. $(w', z')$, to be possible ($z'$ is an alternative of $z$) and in such a pair $\alpha$ holds after the action sequence $z'$ in world $w'$. For example, let $\alpha$ be $h = 0$ and $z = fwd(1,0)$. Since the outcome is selected by the nature, it could be the case that: 1) the outcome of the forward action is actually $fwd(1,0)$ and it was in a world where initially $h = 0$; or 2) the outcome of the forward action is actually $fwd(1,1)$ and it was in a world where initially $h = 1$. Both cases result in $h = 0$. Hence, $\mathcal{W}_\alpha$ is essentially the set of all alternative pairs (world and action sequence) of $z$ that might result in $\alpha$.

A distribution $d$ is **regular** iff EQ$(d, \mathcal{W}^{\{d\}}_{\text{TRUE}}, n)$ for some $n \in \mathbb{R}^{>0}$. In essence, regular distributions are indeed discrete probability distributions. We denote the set of all regular distributions as $\mathcal{D}$.

**Definition 6.** Given $w \in \mathcal{W}, d \in \mathcal{D}, z \in \mathcal{Z}$, we define

- $w_z$ as a world such that for all primitive terms $t$ and $z' \in \mathcal{Z}$, $w_z[t, z'] = w[t, z \cdot z']$;
- $d_z$ as a mapping such that for all $w \in \mathcal{W}$, $d_z(w) = \sum_{\{w':d(w')>0\}} \sum_{\{z':z' \sim_{w'} z\ ,\ (w')_{z'} = w\}} d(w') \times l^*(w', z')$.

$w_z$ is called the **progressed world** of $w$ wrt $z$. Since worlds are tree-structured, the progressed world $w_z$ of a world $w$ wrt an action sequence $z$ is just a copy of the sub-tree of $w$ starting after the action sequence $z$, i.e. *forgetting* the past. Fig. 3 illustrates the progressed world $w_{fwd(1,1)}$ of the possible world $w$ in Fig. 2 wrt the action $fwd(1,1)$. $d_z$ is called the **progressed distribution** of $d$ wrt $z$ and it is obtained by shifting the weight of worlds according to the actions' likelihood.

A remark is that the $d_z$ might not be regular for a regular $d$. For example, if the likelihood of $z$'s alternatives is all zero in worlds with non-zero weights, then $\text{Eq}(d_z, \mathcal{W}_{\text{TRUE}}^{d_z}, 0)$. Hence we define:

**Definition 7.** A distribution $d$ is **compatible** with action sequence $z$, $d \sim_{comp} z$ iff $d_z \in \mathcal{D}$; given an epistemic state $e$, the set $e_z = \{d_z | d \in e \cap \mathcal{D}, d \sim_{comp} z\}$ is called the **progressed epistemic state** of $e$ wrt $z$.

As a consequence, $d \sim_{comp} \langle\rangle$ iff $d \in \mathcal{D}$. Note that the progressed epistemic state of $e$ is only about its regular subset $e \cap \mathcal{D}$ and $e_z \subseteq \mathcal{D}$, therefore $e \neq e_{\langle\rangle}$ in general.

The truth of $\boldsymbol{B}$ and $\boldsymbol{O}$ is given by:

- $e, w, z, \lambda \models \boldsymbol{B}(\alpha : r)$ iff $\forall d \in e_z$, $\text{Norm}(d, \mathcal{W}_{\alpha}^{\{d\},\lambda}, \mathcal{W}_{\text{TRUE}}^{\{d\},\lambda}, n)$ for $n \in \mathbb{R}$ and $n = |r|_\lambda$;
- $e, w, z, \lambda \models \boldsymbol{O}(\alpha_1 : r_1, \ldots, \alpha_k : r_k)$ iff $\forall d$, $d \in e_z$ iff for all $1 \leq i \leq k$, $\text{Norm}(d, \mathcal{W}_{\alpha_i}^{\{d\},\lambda}, \mathcal{W}_{\text{TRUE}}^{\{d\},\lambda}, n_i)$ for $n_i \in \mathbb{R}$, and $n_i = |r_i|_\lambda$.

Intuitively, an epistemic state $e$ only believes something after action sequence $z$, if and only if the progressed epistemic state $e_z$ is the maximal epistemic state consisting of all distributions that satisfy the belief. However, this definition of $\boldsymbol{O}$ is problematic to some extent. The reason is that, in some metric space, the set of distributions that satisfies only-believing is a closed set (defined below) while $e_z$ might not always be closed. Thinking about the moving robot with an initial epistemic state $e$ that satisfies only knows (beliefs with degree 1) its distance to the wall is either 1 or 2, i.e. $h = 1 \vee h = 2$. After the accurate sensing $sonar(2)$, it should only know $h = 2$. Meanwhile, since epistemic states that satisfy $\boldsymbol{O}(h = 2)$ are closed, the updated epistemic state $e_{sonar(2)}$ of $e$ should be closed as well. However, $e_z$ in Definition 7 cannot guarantee this. To see this more formally, we need the following notations.

Informally, a metric space $(M, \rho)$ is a set $M$ with a metric $\rho$ associated with it. A typical example of metric space is the $n$-dimensional Euclidean space $\mathbb{R}^n$ with the Euclidean distance as a metric. Specifically, let $S(d)$ be the support set of $d$, i.e. $S(d) = \{w : d(w) > 0\}$, we define the distance function $\rho$ for regular distributions $\mathcal{D}$ as $\rho(d, d') = \sum_{w \in S(d) \cup S(d')} |d(w) - d'(w)|$.

**Proposition 1.** *$(\mathcal{D}, \rho)$ forms a metric space (proof in Appendix A.1).*

Given an infinite sequence of regular distributions $\{d_1, d_2, \ldots d_i, \ldots\}$, we say it converges to a regular distribution $d$ if for any $\epsilon > 0$, there exists a natural number $N \in \mathbb{N}$ s.t. for any $i > N$, $\rho(d_i, d) < \epsilon$. If such a $d$ exists, we write $\lim_{i\to\infty} d_i = d$ and call $d$ the limit of $\{d_1, d_2, \ldots d_i, \ldots\}$.

**Definition 8** *(Closure).* For all $e \subseteq \mathcal{D}$, the closure of $e$ is defined as $cl(e) = \{d | d \in \mathcal{D}, \exists\{d_1, d_2, \ldots d_i, \ldots\}, \forall i \in \mathbb{N}, d_i \in e, \lim_{i\to\infty} d_i = d\}$.

We call an $e \subseteq \mathcal{D}$ closed if and only if $e = cl(e)$.

**Proposition 2.** *Given a regular $d$ and sequence $\{d_1, d_2, \ldots d_i, \ldots\}$ s.t. $d = \lim_{i\to\infty} d_i$, for any $w \in \mathcal{W}$, the limit of $d_i(w)$ exists and $d(w) = \lim_{i\to\infty} d_i(w)$.*

The proof is trivial since $\rho(d, d_i) \geq |d(w) - d_i(w)|$ for any $d, d_i, w$.

**Theorem 1.** *Let $e$ be an epistemic state, $\lambda$ a variable map, $\alpha$ an objective sentence, and $r$ a rigid term, if for all $d \in e$, $\text{Norm}(d, \mathcal{W}_{\alpha}^{\{d\},\lambda}, \mathcal{W}_{\text{TRUE}}, |r|_\lambda)$, then for all $d' \in cl(e)$, $\text{Norm}(d', \mathcal{W}_{\alpha}^{\{d'\},\lambda}, \mathcal{W}_{\text{TRUE}}, |r|_\lambda)$ (proof in Appendix A.2).*

The theorem suggests that $\{d | \text{Norm}(d, \mathcal{W}_{\alpha}^{\{d\},\lambda}, \mathcal{W}_{\text{TRUE}}, |r|_\lambda)\}$ is a closed set for any objective $\alpha, \lambda$, and $r$. Therefore, $e_z$ should be closed to enable the maximal semantics for only-believing. Hence, we have the following definition:

**Definition 9** *(Progressed epistemic state).* Given $e$, $z \in \mathcal{Z}$, we define $e_z$ as $e_z = cl(\{d_z \in \mathcal{D} : d \in e \cap \mathcal{D}, d \sim_{comp} z\})$, i.e. $e_z = \{d \in \mathcal{D} : \exists\{d_1, d_2, ..., d_i, ...\}, \forall i \in \mathbb{N}, d_i \in e \cap \mathcal{D}, d_i \sim_{comp} z, d = \lim_{i\to\infty}(d_i)_z\}$.

In the conference version of this work [1], the question whether the $e_z$ in Definition 7 already ensures that $e_z$ is closed remained open. If the answer were positive, i.e. $e_z$ in Definition 7 is already closed, there is no need to explicitly impose the closure operator $cl(\cdot)$, hence simplifying the semantics. In this paper, we prove that the answer is indeed negative, namely, there exists a closed $e$ where the set $\{d_z | d \in e \cap \mathcal{D}, d \sim_{comp} z\}$ is not closed. An example can be found in Section 5. Now, the truth of $\boldsymbol{B}$ and $\boldsymbol{O}$ are given exactly the same as before except $e_z$ with the new definition.

For a sentence $\alpha$, we write $e, w \models \alpha$ to mean $e, w, \langle\rangle, \lambda \models \alpha$ for all variable maps $\lambda$. When $\Sigma$ is a set of sentences and $\alpha$ is a sentence, we write $\Sigma \models \alpha$ (read: $\Sigma$ logically entails $\alpha$) to mean that for every set of regular distributions $e$ and $w$, if

$e, w \models \alpha'$ for every $\alpha' \in \Sigma$, then $e, w \models \alpha$. We say that $\alpha$ is valid ($\models \alpha$) if $\{\} \models \alpha$. Satisfiability is then defined in the usual way. If $\alpha$ is an objective formula, we write $w \models \alpha$ instead of $e, w \models \alpha$. Similarly, we write $e \models \alpha$ instead of $e, w \models \alpha$ if $\alpha$ is subjective.

### 2.3. Comparison with $\mathcal{DS}$ and some properties

Now, let us compare the first-order fragment of $\mathcal{DS}_p$ with $\mathcal{DS}$ (recall that $\mathcal{DS}$ is first-order). The languages of $\mathcal{DS}$ and $\mathcal{DS}_p$ are rather similar except that $\mathcal{DS}$ considers fluent predicates while $\mathcal{DS}_p$ only has fluent functions. While in $\mathcal{DS}$ every closed term is a standard name, we follow the work of Lakemeyer and Levesque [11], which uses special standard names for objects and actions. The semantic structures in both logics are essentially the same, consisting of worlds, sets of distributions over worlds serving as epistemic states, and action sequences. Our use of rigid mathematical functions, which are not considered in $\mathcal{DS}$, is similar to the $\mathcal{R}$-interpretation in [12]. Among other things, this allows us later to express degrees of belief specified by arbitrary rigid terms.

The main difference lies in the semantics of beliefs and only-believing. To appreciate the difference, it is instructive to review the semantics of $\mathcal{DS}$. While many notations are exactly the same, things diverge in the definition of $\mathcal{W}_\alpha$. First, $\mathcal{DS}$ keeps the traditional special *poss* predicate to specify action preconditions, and a notion of $exec(z)$, defined recursively by *poss*, to express that action sequence $z$ is executable. Second, $\mathcal{DS}$ defines a compatibility between worlds wrt $oi$: $w' \sim_{oi} w$ iff for all $a, a'$, and $z$, $w'[oi(a,a'), z] = w[oi(a,a'), z]$, that is, $w'$ is compatible with $w$ if they agree on $oi$. Finally, $\mathcal{W}_\alpha$ in $\mathcal{DS}$ is then defined wrt triples $e, w, z$[4]: $\mathcal{W}_\alpha^{e,w,z} = \{(w', z') : z' \sim_{w'} z, w' \sim_{oi} w, \text{ and } e, w', \langle\rangle \models [z'] \wedge exec(z')\}$. Note that worlds in $\mathcal{W}_\alpha^{e,w,z}$ agree with $w$ wrt observational-indistinguishability. Truth conditions of beliefs ($\boldsymbol{B}'$) and only-believing ($\boldsymbol{O}'$) are given by

- $e, w, z \models \boldsymbol{B}'(\alpha : r)$ iff $\forall d \in e$ (not $e_z$), $\textsc{Norm}(d, \mathcal{W}_\alpha^{e,w,z}, \mathcal{W}_{\text{TRUE}}, r)$ for *constant* $r$;
- $e, w, z \models \boldsymbol{O}'(\alpha_1 : r_1, \ldots, \alpha_k : r_k)$ iff $\forall d$, $d \in e$ iff $\forall i.1 \leq i \leq k$, $\textsc{Norm}(d, \mathcal{W}_{\alpha_i}^{e,w,z}, \mathcal{W}_{\text{TRUE}}, r_i)$ for *constants* $n_i$.

One consequence of only allowing constants in degrees of belief is that a formula like $\boldsymbol{B}(p : 0.1 + 0.2)$ is not well-defined. The new logic overcomes this by a special treatment of rigid terms, that is, terms like $0.1 + 0.2$ have the same denotations in all worlds, i.e. 0.3. This, among other things, enables us to include formulas like $\forall u.\boldsymbol{B}(h = u : \mathcal{G}(\frac{1}{2}, u))$ as well-formed formulas, where $h$ is a fluent and $\mathcal{G}(\frac{1}{2}, u)$ stands for geometric distribution with expectation 2.

Another observation is that, while $\mathcal{W}_\alpha$ in $\boldsymbol{B}'$ and $\boldsymbol{O}'$ involves $e$ and $w$, $\mathcal{W}_\alpha$ in $\boldsymbol{B}$ and $\boldsymbol{O}$, however, only involves individual $d \in e$. Such a change has major impacts on properties of logic like introspection and meta-beliefs. Discussing them would go beyond the scope of this paper, see [13] for a detailed discussion. Nevertheless, the following is a unique property of our logic (hence not in [13]): let the set of (first-order) formulas of *strictly positive belief* (or simply strictly positive beliefs) be recursively defined as:

- for any objective formula $\alpha_{obj}$ and rigid term $r$, $\boldsymbol{B}(\alpha_{obj} : r)$ is a formula of strictly positive belief;
- if $\alpha, \alpha_1, \alpha_2$ are positive beliefs, then $\boldsymbol{K}\alpha$, $\alpha_1 \wedge \alpha_2$, $\alpha_1 \vee \alpha_2$, $\forall v.\alpha$ are strictly positive beliefs as well.

For example, $\boldsymbol{B}(p : 0.3)$, $\boldsymbol{KB}(p : 0.3)$, $\boldsymbol{B}(p : 0.3) \vee \boldsymbol{B}(p : 0.4)$, $\boldsymbol{B}(p : 0.3) \wedge \boldsymbol{B}(q : 0.3)$, $\forall u.\boldsymbol{B}(h = u : \mathcal{G}(\frac{1}{2}, u))$ are strictly positive beliefs, yet $\boldsymbol{K}\neg\boldsymbol{K}p$ is not.

**Theorem 2.** *For any formula $\alpha$ of strictly positive beliefs, $\boldsymbol{O}\alpha$ is satisfiable.*

The proof is by induction on the structure of strictly positive beliefs and the base case is a consequence of Theorem 1 (see Appendix A.3 for proof).[5]

Meanwhile, the property is unique here and not in [13] due to the use of the closure operator in the semantics of $\boldsymbol{O}$. We defer the discussion on the consequences of the use of the closure operator to Section 5.

Lastly, our semantics of $\boldsymbol{B}$ and $\boldsymbol{O}$ after action sequences $z$ are evaluated against the updated progressed epistemic state $e_z$, while $\boldsymbol{B}'$ and $\boldsymbol{O}'$ in $\mathcal{DS}$ are evaluated against the initial epistemic state $e$, hence $\mathcal{DS}$ lacks a semantics for progression. Specifically, it is unclear what an epistemic state that satisfies $[sonar(2)]\boldsymbol{O}'(h = 2 \wedge \Sigma : 1)$ looks like as aforementioned in the Introduction.

We comment that the idea of a special treatment of rigid terms and using individual $d \in e$ in $\mathcal{W}_\alpha$ for $\boldsymbol{B}$ and $\boldsymbol{O}$ is from Liu and Lakemeyer [13].

**Theorem 3.** *Given an objective FO sentence $\alpha$ and constant $n$,*

- $\models \boldsymbol{B}(\alpha : n) \equiv \boldsymbol{B}'(\alpha : n)$;

[4] Since $\mathcal{DS}$ is first-order, the variable map $\lambda$ is not required.

[5] This is in contrast with the notion of only-believing in [13] where only-believing everything is uniquely satisfiable.

- $\models \boldsymbol{O}'(\alpha : n) \supset \boldsymbol{O}(\alpha : n)$.

Intuitively, in the static case, when $\alpha$ is objective, *poss*, *e*, and $\{d\}$ in $\mathcal{W}_\alpha$ play no role, therefore $\mathcal{W}_\alpha$ coincides in the two logics. Hence $\boldsymbol{B}$ and $\boldsymbol{B}'$ are equivalent. As for only-believing, suppose an $e \subseteq \mathcal{D}$ s.t. $e \models \boldsymbol{O}'(\alpha : n)$, we have $e = \{d : \textsc{Norm}(d, \mathcal{W}_\alpha, \mathcal{W}_{\text{TRUE}}, n)\}$. By Theorem 1, $e$ is closed, therefore, $e = e_{\langle\rangle}$. By the semantics, $e \models \boldsymbol{O}(\alpha : n)$. The converse does not hold since there might exist an open set $e$ s.t. $e \models \boldsymbol{O}(\alpha : n)$, yet $e \neq \{d : \textsc{Norm}(d, \mathcal{W}_\alpha, \mathcal{W}_{\text{TRUE}}, n)\} = cl(e)$ and therefore $e \not\models \boldsymbol{O}'(\alpha : n)$.

Although our notion of only-believing is somewhat weaker than its counterpart of $\mathcal{DS}$, we still retain the following properties of only-believing (Item 1-2 below):

**Proposition 3.** *Let $\alpha$ and $\alpha_i$ be arbitrary FO sentences, we have*

1. $\models \boldsymbol{O}(\alpha_1 : r_1; \ldots \alpha_k : r_k) \supset \bigwedge \boldsymbol{B}(\alpha_i : r_i)$;
2. $\models \boldsymbol{O}(\alpha : r) \supset \neg \boldsymbol{B}(h(\vec{n}) = m : r')$ *for all* $r, r'$, *and* $\alpha$, *where* $\vec{n}$ *and* $m$ *are standard names and* $h$ *is a fluent not in* $\alpha$;
3. *For any* $e$, $e \models \boldsymbol{O}(\alpha_1 : r_1, \ldots, \alpha_k : r_k)$ *iff* $cl(e) \models \boldsymbol{O}(\alpha_1 : r_1, \ldots, \alpha_k : r_k)$

The second part says that the agent has no beliefs about things not mentioned in the KB. Note that this is not true if $\boldsymbol{O}$ is replaced by $\boldsymbol{B}$. The third item is a direct result of Theorem 1. In the rest of the paper, whenever we write $e \models \boldsymbol{O}(\alpha_1 : r_1, \ldots, \alpha_k : r_k)$, we mean a closed $e$, unless stated otherwise.

Before we move on, let us turn back to the motivational example in the introduction. In the rest of the paper, we will use some conventions to simplify formulas including that free variables are implicitly universal quantified outside. The modality $\Box$ has lower syntactic precedence than the connectives, and $[\cdot]$ has the highest priority.

**Example 1.** In the logic $\mathcal{DS}$, suppose $\Sigma$ is the conjunction of the following

1. $\Box poss(a) = \text{TRUE}$
2. $\Box oi(a, a') = \top \equiv \exists y. a = sonar(y) \wedge a = a'$
3. $\Box l(a) = x \equiv x = 1 \wedge \exists y. h = y \wedge a = sonar(y)$

then $\not\models \boldsymbol{O}'(h = 1 : 0.5,\ h = 2 : 0.5,\ \Sigma : 1) \supset [sonar(2)]\boldsymbol{O}'(h = 2 \wedge \Sigma : 1)$.

Suppose the opposite holds and $e, w \models \boldsymbol{O}'(h = 1 : 0.5,\ h = 2 : 0.5,\ \Sigma : 1)$. Consider a distribution $d$ s.t. $\{d\}, w \models \boldsymbol{K}'(h = 1 \wedge \Sigma : 1)$, clearly, $d \notin e$ since $e, w \models \boldsymbol{B}(h = 1 : 0.5)$. In addition, one can check $\{d\}, w \models [sonar(2)]\boldsymbol{K}'(h = 1 \wedge \Sigma : 1)$. However, by hypothesis, $e, w \models [sonar(2)]\boldsymbol{O}'(h = 2 \wedge \Sigma : 1)$. By semantics of $\boldsymbol{O}'$, $d \in e$, contradiction. The reason is that the truth of $\boldsymbol{O}'$ refers to $e$ but not the progressed epistemic state $e_z$.

**Lemma 1.** $(w_z)_{z'} = w_{z \cdot z'}$ *and* $(e_z)_{z'} = e_{z \cdot z'}$ *(proof in Appendix A.5).*

The proof of the first is trivial by the definition of $w_z$ while the proof of the latter requires not just the definition of $e_z$ but also the triangle inequality property of the distance function $\rho$.

**Theorem 4.** *For any FO sentence* $\psi$, $e, w, z \models \psi$ *iff* $e_z, w_z \models \psi$.

**Proof. Base**: For atomic formula $t_1 = t_2$, $e, w, z \models t_1 = t_2$, iff by semantics $w[t_1, z] = w[t_2, z]$, iff by Def. of $w_z$ $w_z[t_1, \langle\rangle] = w_z[t_2, \langle\rangle]$, iff by semantics $e_z, w_z \models t_1 = t_2$.

**Induction**: The induction of connectives $(\neg, \wedge, \forall)$ is trivially by semantics.

- For $\boldsymbol{B}$ (and similarly for $\boldsymbol{O}$):
  $e, z \models \boldsymbol{B}(\phi : r)$ iff (by semantics) $\forall d \in e_z, \textsc{Norm}(d, \mathcal{W}_\phi^{\{d\}}, \mathcal{W}_{\text{TRUE}}^{\{d\}}, |r|)$ iff (by semantics) $e_z \models \boldsymbol{B}(\phi : r)$
- For $[\cdot]$ (and similarly for $\Box$, since $\psi$ is FO, we use $|t_a|_w^z$ for the denotation of term $t_a$ under $w, z$):
  $e, w, z \models [t_a]\psi$ iff (by semantics) $e, w, z \cdot n, \models \psi$ for $n = |t_a|_w^z$ iff (by induction)
  $e_{z \cdot n}, w_{z \cdot n} \models \psi$ for $n = |t_a|_w^z$ iff (by Lemma. 1) $(e_z)_n, (w_z)_n \models \psi$ for $n = |t_a|_w^z$ iff (by induction)
  $e_z, w_z, n \models \psi$ for $n = |t_a|_w^z$ iff (by Def. of $w_z$) $e_z, w_z, n \models \psi$ for $n = |t_a|_{w_z}^{\langle\rangle}$ iff (by semantics) $e_z, w_z \models [t_a]\psi$. $\Box$

## 3. The semantics of progression

Sometimes, it would be desirable to include mathematical functions beyond $+, \times$ as logical terms like the uniform distribution $\mathbf{U}_{\{1,2\}}$ mentioned earlier. Specially, we might use summation $\sum$ as a logical term.[6] One way is to assume infinite rigid function symbols, one for each such function, and semantically ensure all worlds in $\mathcal{W}$ interpret them identically. Another way is to specify them syntactically by axioms. Here, we take the latter approach and call these axioms *definitional axioms*,[7] these functions *definitional functions*, and terms constructed by definitional functions *definitional terms*. E.g. the following axiom specifies $\mathbf{U}_{\{1,2\}}$.

$$\forall v.\forall u.\mathbf{U}_{\{1,2\}}(u) = v \equiv (u = 1 \vee u = 2) \wedge v = 0.5 \vee \neg(u = 1 \vee u = 2) \wedge v = 0 \tag{1}$$

### 3.1. Basic action theories

BATs were first introduced by Reiter [16] to encode the effects and preconditions of actions. Given a finite set of fluents $\mathcal{H}$, a BAT $\Sigma$ over $\mathcal{H}$ consists of the union of the following sets:

- $\Sigma_{ssa}$: A set of successor state axioms, one for each fluent $h$ in $\mathcal{H}$, of the form $\square[a]h(\vec{p}) = u \equiv \gamma_h$ to characterize action effects, also providing a solution to the frame problem [16]. Here $\gamma_h$ is a fluent formula with free variables $\vec{p}, u$ and it is functional in $u$.
- $\Sigma_{oi}$: A single observation-indistinguishability axiom of the form $\square oi(a, a') = \top \equiv \psi$, where $\psi$ is a rigid formula, to specify whether two actions are mutually alternative.
- $\Sigma_l$: A set of likelihood axioms of the form $\square l(a) = \mathcal{L}(a)$ to capture action likelihoods. Here $\mathcal{L}(a)$ is a definitional term with free variable $a$.

We require $\Sigma_{ssa}$, $\Sigma_{oi}$, and $\Sigma_l$ to be first-order. The condition that $\psi$ is rigid ensures that the observational-indistinguishability among actions is fixed. Besides BATs, to infer future states, we need to specify what holds initially. This is achieved by a set of fluent sentences $\Sigma_0$ (which may be second-order). By **belief distribution**, we mean the joint distribution of a finite set of random variables. Formally, assuming all fluents in $\mathcal{H}$ are **nullary**,[8] $\mathcal{H} = \{h_1, \dots h_k\}$, a belief distribution $\boldsymbol{B}^f$ of $\mathcal{H}$ is a formula of the form $\forall \vec{u}.\boldsymbol{B}(\vec{h} = \vec{u} : f(\vec{u}))$, where $\vec{u}$ is a set of variables, $\vec{h} = \vec{u}$ stands for $\bigwedge h_i = u_i$, $f$ is a definitional function of sort number with free variables $\vec{u}$. Finally, by a knowledge base, we mean a sentence of the form $\boldsymbol{O}(\boldsymbol{B}^f \wedge \Sigma)$. Although the logic allows KB specified by multiple distributions, for example, $\boldsymbol{O}(\boldsymbol{B}^{f_1} \vee \boldsymbol{B}^{f_2})$, we only consider KB with a single distribution in this paper.

**Example 2.** The following is a possible BAT $\Sigma$ for our robot moving example:

$$\square[a]h = u \equiv \exists x, y.a = fwd(x, y) \wedge u = \max\{0, h - y\} \ \vee \forall x, y.a \neq fwd(x, y) \wedge h = u$$

$$\square oi(a, a') = \top \equiv \exists x, y, z.a = fwd(x, y) \wedge a' = fwd(x, z) \vee a = sonar(z) \wedge a' = a$$

$$\square l(a) = \mathcal{L}(a) \text{ with } \mathcal{L}(a) = \begin{cases} \theta(x, y, 0.2, 0.6) & \exists x.\exists y.a = fwd(x, y) \\ \theta(z, h, 0.1, 0.8) & \exists z.a = sonar(z) \end{cases}$$

and further more, $\theta$ is given by

$$\theta(x, y, m, n) = \begin{cases} m & |x - y| = 1 \\ n & x = y \\ 0 & o.w. \end{cases}$$

A possible KB is $\boldsymbol{O}(\boldsymbol{B}^f \wedge \Sigma)$ where $f$ is given by[9] $f(u) = \begin{cases} 1/3 & u \in \{1, 2, 3\} \\ 0 & o.w. \end{cases}$

In English, distance $h$ can only be affected by $fwd(x, y)$ and the value is determined by value $y$, not the intended value $x$; the robot cannot get across the wall ($u = \max\ \{0, h - y\}$); two actions are observationally indistinguishable if and only if they are both forward actions with the same intended value or they are identical sensing action; likelihood of stochastic action $fwd(x, y)$ and noisy sensing $sonar(z)$ is specified by $\theta$. The agent considers a uniform distribution among $\{1, 2, 3\}$ initially.

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[6] Summation is second-order definable. See [12] for details. A problem of summation as a logical term is that summation is not closed under the computable domain since the limit of a infinite summation of rational could be a non-computable number [14], see also *Specker Sequence* [15]. Hence, for some terms with infinite summations, we cannot assign decent denotations. We use a special reserved standard name *undefined* for this purpose.

[7] In the rest of paper, whenever we write logical implication $\Sigma \models \alpha$, we implicitly mean $\Sigma \wedge \Delta \models \alpha$, where $\Delta$ is the set of all definitional axioms of functions involved in $\Sigma$ and $\alpha$.

[8] As discussed in [12], allowing fluents with arguments would result in joint distribution over infinitely many random variables, which is generally problematic in probability theory.

[9] Here, "$\in$" should be understood as a finite disjunction. For readability, we write the definition functions in this form, they should be understood as logical formula as Equation (1).

### 3.2. Definition of progression

For the objective fragment, our definition of progression is similar to [17]:

**Definition 10.** Given $\Sigma_0$, $\Sigma$, and a rigid ground action term $t$, a set of fluent formulas $\Sigma_0'$ is called the progression of $\Sigma_0$ wrt $t, \Sigma$ iff for all $w'$, $w' \models \Sigma_0' \wedge \Sigma$ iff there exists $w$ s.t. $w \models \Sigma_0 \wedge \Sigma$ and $w_{|t|} = w'$.

**Theorem 5** *(Lin and Reiter style of progression). The following is a progression of $\Sigma_0$ wrt $\Sigma, t$:*

$$\exists \vec{F}.(\Sigma_0)^{\mathcal{H}}_{\vec{F}} \bigwedge \forall \vec{p}.\forall u.h(\vec{p}) = u \equiv (\gamma_h)^{a,\mathcal{H}}_{t,\vec{F}}$$

*where $\vec{F}$ are SO variables s.t. $F_i$, $h_i$ are of the same arity. Here $(\Sigma_0)^{\mathcal{H}}_{\vec{F}}$ is the formula obtained by replacing all fluents $\mathcal{H}$ in $\Sigma_0$ with second-order variables $\vec{F}$.*

Since the LR (Lin and Reiter's) progression was proposed, efforts have been made to find fragments where $\Sigma_0'$ is first-order. For example, if the successor state of axioms has *local-effect* [18], the progression is first-order definable. Intuitively, local-effect means actions can only affect locally, like the block world example, the action $move(x, y, z)$, i.e. moving object $x$ from $y$ to $z$, only has local-effect on the object $x$ and location $y$ and $z$ but nothing globally. Here we show that the progression of the **nullary fluent** fragment is first-order definable. Henceforth, we use $\alpha^{\vec{h}}_{\vec{v}}$ to mean the formula obtained by replacing all nullary fluents $\vec{h}$ in $\alpha$ with free variables $\vec{v}$.

**Theorem 6.** *Given $\Sigma_0$, $\Sigma$, and an ground action term $t$ where every fluent in $\mathcal{H}$ is nullary, the following is a progression of $\Sigma_0$ wrt $\Sigma, t$:*

$$\exists \vec{v}.(\Sigma_0)^{\vec{h}}_{\vec{v}} \bigwedge \forall u.h = u \equiv (\gamma_h)^{a,\vec{h}}_{t,\vec{v}}.$$

The proof is based on the fact that $\exists V.\alpha$ is logically equivalent to $\exists v.(\alpha)^V_v$ if $V$ is nullary (see Appendix A.6). We mark the FO progression given by Theorem 6 as $Pro(\Sigma_0, \Sigma, t)$, when the context is clear we write $Pro(\Sigma_0, t)$. For example, let $\Sigma_0 = \{h = 1 \vee h = 2\}$, then $Pro(\Sigma_0, \Sigma, fwd(1, 2)) = \{\exists v.(v = 1 \vee v = 2) \wedge \forall u.h = u \equiv u = \max\{0, v - 2\}\} = \{h = 0\}$.

There is a comment in [19] which claims that

> "... because we are assuming a finite set of nullary fluents, any basic action theory can be shown to be local-effect, where progression is first-order definable."

Yet, this is incorrect since we can construct a BAT with only nullary fluents to simulate a two-counter machine as in [20] whose $\Sigma_{ssa}$ is not local-effect.

For the subjective fragment, as mentioned in the introduction, the progression of a probabilistic knowledge base should correspond to what the agent only-believes after actions. Formally, we have

**Definition 11.** We call a formula $\boldsymbol{O}(\Psi \wedge \Sigma)$ the progression of the knowledge base $\boldsymbol{O}(\boldsymbol{B}^f \wedge \Sigma)$ wrt ground action $t$, if and only if $\boldsymbol{O}(\boldsymbol{B}^f \wedge \Sigma) \models [t]\boldsymbol{O}(\Psi \wedge \Sigma)$.

Essentially, the task of progression is to find such a $\Psi$ that follows from $\boldsymbol{B}^f$.

### 3.3. Progression after sensing actions

To begin with, it's necessary to define what sensing actions are. Our view of sensing actions is the same as in [8], namely these are actions that provide information about the world but do not change it. In a formal way, they are actions that appear in the likelihood axioms but not in the successor state axioms, like *sonar* in Example 2. Additionally, sensing actions have no alternatives. Intuitively, this means that when the sonar reads a value, the agent knows it reads that value but no others.

**Theorem 7.** *Given a KB $\boldsymbol{O}(\boldsymbol{B}^f \wedge \Sigma)$ and a sensing action $t_{sen}$ s.t. $\boldsymbol{O}(\boldsymbol{B}^f \wedge \Sigma) \models \boldsymbol{K}(l(t_{sen}) > 0)$, then $\boldsymbol{O}(\boldsymbol{B}^f \wedge \Sigma) \models [t_{sen}]\boldsymbol{O}(\boldsymbol{B}^{f'} \wedge \Sigma)$ where $f'$ is a definitional function in term of $f$ as:*

$$f'(\vec{u}) = \frac{1}{\eta} f(\vec{u}) \times \mathcal{L}(t_{sen})^{\vec{h}}_{\vec{u}}, \text{ and } \eta = \sum_{\vec{u}' \in (\mathcal{N}_O)^k} f(\vec{u}') \times \mathcal{L}(t_{sen})^{\vec{h}}_{\vec{u}'}.$$

Namely, the progression of a KB is another KB with belief distribution $\boldsymbol{B}^{f'}$; the relation between $\boldsymbol{B}^f$ and $\boldsymbol{B}^{f'}$ is such that the new degree of belief of $(\vec{h} = \vec{u})$ is just the normalized product of the old degree of belief and likelihoods of the sensing action. While intuitively the result might be straightforward, the proof is non-trivial. Suppose $e \models \boldsymbol{O}(\boldsymbol{B}^f \wedge \Sigma)$ and $e' \models \boldsymbol{O}(\boldsymbol{B}^{f'} \wedge \Sigma)$, the central task of the proof is to show that $e_{t_{sen}} = e'$. The direction $e_{t_{sen}} \subseteq e'$ is straightforward, whereas the other one is sophisticated. In fact, for the direction $e_{t_{sen}} \subseteq e'$, we have:

**Lemma 2.** *$\boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma \models [t_{sen}]\boldsymbol{B}^{f'}$, where $\boldsymbol{B}^f$, $\Sigma$, $\boldsymbol{B}^{f'}$, and $t_{sen}$ are the same as in Theorem 7 (see Appendix A.7 for proof).*

**Definition 12.** Given BAT $\Sigma$ wrt fluents $\mathcal{H}$, let $\mathcal{P}_{\bar{\mathcal{H}}}$ be the set of all primitive terms of fluents not in $\mathcal{H}$, we define a relation $\simeq_{\mathcal{H}}$ over $\mathcal{W}$ as $w \simeq_{\mathcal{H}} w'$ iff for all $t \in \mathcal{P}_{\bar{\mathcal{H}}}$ and all $z \in \mathcal{Z}$, $w[t, z] = w'[t, z]$.

Namely, $w \simeq_{\mathcal{H}} w'$ iff $w$ and $w'$ assign the same denotation for terms without fluents in $\mathcal{H}$. Clearly, $\simeq_{\mathcal{H}}$ is an equivalence relation. We denote the set of all equivalence classes wrt BAT $\Sigma$ as $\mathcal{W}_{\mathcal{H}}$. Note that $\mathcal{W}_{\mathcal{H}}$ is uncountable.

**Proposition 4.** *Given BAT $\Sigma$ and $\mathcal{C} \in \mathcal{W}_{\mathcal{H}}$, for all standard names $\vec{n}$, there is a unique world $w$ s.t. $w \in \mathcal{C}$ and $w \models \vec{h} = \vec{n} \wedge \Sigma$, we mark this world as $w_{\mathcal{C},\vec{n}}$.*

This is straightforward as the condition $w \models \Sigma \wedge \vec{h} = \vec{n}$ restricts how $w$ interprets terms involving fluents in $\mathcal{H}$ and the condition $w \in \mathcal{C}$ restricts how $w$ interprets all other terms (terms in $\mathcal{P}_{\bar{\mathcal{H}}}$).

**Lemma 3.** *Let $\boldsymbol{B}^f$, $\boldsymbol{B}^{f'}$, $\Sigma$, and $t_{sen}$ be as in Theorem 7. For all $d' \in \mathcal{D}$ such that $\{d'\} \models \boldsymbol{B}^{f'} \wedge \boldsymbol{K}\Sigma$, there exists a distribution $d \in \mathcal{D}$ s.t. $\{d\} \models \boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma$ and $d_{t_{sen}} = d'$.*

The construction of such $d$ is based on three main steps:

1. By virtue of Proposition 4 and the fact that $S(d')$ is countable, there exists a minimal countable set $\mathcal{W}_{cov}(d') \subseteq \mathcal{W}_{\mathcal{H}}$ such that $\forall w \in S(d'), \exists \mathcal{C}' \in \mathcal{W}_{cov}(d'), w \in \mathcal{C}'$, namely $\mathcal{W}_{cov}(d')$ covers all worlds in $S(d')$;
2. For each $\mathcal{C}' \in \mathcal{W}_{cov}(d')$, we select a $\mathcal{C} \in \mathcal{W}_{\mathcal{H}}$ such that for every world $w' \in \mathcal{C}'$, there exists a unique world $w \in \mathcal{C}$ which can progress to $w'$ after $t_{sen}$. The selected $\mathcal{C}$ forms $\mathcal{W}_{cov}(d)$;
3. The last step is to assign weights to $w$ which is exactly the weight of $w'$ under $d'$ divided by the likelihood of $t_{sen}$.

Essentially, $f(\vec{n}) = f'(\vec{n})/\mathcal{L}(t_{sen})^{\vec{h}}_{\vec{n}}$ if we ignore the $\eta$ in Theorem 7. Our construction of $d$ is to reconstruct such relation at the semantic level: If $d'$ assigns some weights to a world $w'$, then $d$ assigns a world $w$, which progresses to $w'$ after $t_{sen}$, with the same weights but divided by the likelihood of $t_{sen}$. Moreover, the nature of sensing actions ensures that this semantical property can be reflected correctly at the syntactical level (Detailed in Appendix A.8).

**Proof of Theorem 7.** Suppose two closed $e, e'$ s.t. $e \models \boldsymbol{O}(\boldsymbol{B}^f \wedge \Sigma)$ and $e' \models \boldsymbol{O}(\boldsymbol{B}^{f'} \wedge \Sigma)$, by Theorem 4, it suffices to show $e_{t_{sen}} = e'$

"$e_{t_{sen}} \subseteq e'$": Since $\boldsymbol{O}(\boldsymbol{B}^f \wedge \Sigma) \models \boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma$, $e \models \boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma$ by hypothesis. By Lemma 2 and Theorem 4, $e_{t_{sen}} \models \boldsymbol{B}^{f'}$. By semantics of $\boldsymbol{O}$ and hypothesis, $e_{t_{sen}} \subseteq e'$

"$e' \subseteq e_{t_{sen}}$": Since $e' \models \boldsymbol{O}(\boldsymbol{B}^{f'} \wedge \boldsymbol{K}\Sigma)$, for all $d' \in e'$, $\{d'\} \models \boldsymbol{B}^{f'} \wedge \boldsymbol{K}\Sigma$. By Lemma 3, $\exists d.\{d\} \models \boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma$ and $d_{t_{sen}} = d'$. Since $e \models \boldsymbol{O}(\boldsymbol{B}^f \wedge \Sigma)$ by hypothesis, $d \in e$. $d \sim_{comp} t_{sen}$ due to $d_{t_{sen}} = d'$. Therefore $d' \in \{d \in \mathcal{D} : d \in e \wedge d \sim_{comp} t_{sen}\}$, hence $d' \in e_{t_{sen}}$. That is $e' \subseteq e_{t_{sen}}$. □

**Example 3.** Given KB $\boldsymbol{O}(\boldsymbol{B}^f \wedge \Sigma)$ from Example 2, then $\boldsymbol{O}(\boldsymbol{B}^f \wedge \Sigma) \models [sonar(2)]\boldsymbol{O}(\boldsymbol{B}^{f'} \wedge \Sigma)$, where $f'$ is a definitional function defined as:

$$f'(u) = \frac{1}{\eta} f(u)\mathcal{L}(sonar(2))^h_u = \frac{1}{\eta} f(u)\theta(2, u, 0.1, 0.8)$$

$$= \begin{cases} \frac{1}{\eta}(\frac{1}{3} \times 0.1) & u \in \{1, 3\} \\ \frac{1}{\eta}(\frac{1}{3} \times 0.8) & u = 2 \\ 0 & o.w. \end{cases} = \begin{cases} 0.1 & u \in \{1, 3\} \\ 0.8 & u = 2 \\ 0 & o.w. \end{cases}$$

The second equality is by the specification of $\mathcal{L}(a)$, the third equality is because $f(u)$ is non-zero only among $\{1, 2, 3\}$. The last one is because $\eta = \frac{1}{3}$.

### 3.4. Progression after stochastic actions

Unlike sensing, stochastic actions have observationally indistinguishable actions as alternatives, sometimes even infinitely many alternatives. Besides, stochastic actions do affect the real world. This makes the progression wrt stochastic actions more complicated than sensing actions.

**Theorem 8.** *Given a KB $\boldsymbol{O}(\boldsymbol{B}^f \wedge \Sigma)$ and a stochastic action $t_{sa}$, $\boldsymbol{O}(\boldsymbol{B}^f \wedge \Sigma) \models [t_{sa}]\boldsymbol{O}(\boldsymbol{B}^{f'} \wedge \Sigma)$, where $f'$ is the following definitional function:*

*$f'(\vec{u}) = \sum_{\vec{u}' \in (\mathcal{N}_O)^k} \sum_{a \in \mathcal{N}_A} f(\vec{u}') \times \mathcal{L}(a)^{\vec{h}}_{\vec{u}'} \times \mathbb{I}(\vec{u}, \vec{u}', a, t_{sa})$ where $\mathbb{I}$ is a definitional function given by*

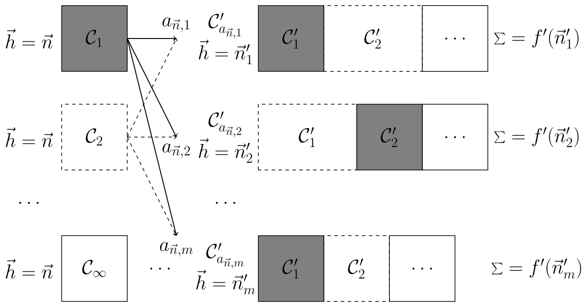

**Fig. 4.** The selection of $S(d)$ for stochastic actions with finite alternatives.

$$\mathbb{I}(\vec{u},\vec{u}',a,t_{sa}) = \begin{cases} 1 & Pro(\vec{h}=\vec{u}',a)^{\vec{h}}_{\vec{u}} \wedge (\psi)^{a'}_{t_{sa}} \\ 0 & o.w. \end{cases}$$

Like sensing actions, the central task of the proof is to show $e_{t_{sa}} = e'$ and the direction $e_{t_{sa}} \subseteq e'$ is straightforward. Formally, we have

**Lemma 4.** *$\boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma \models [t_{sa}]\boldsymbol{B}^{f'}$, where $\boldsymbol{B}^f$, $\Sigma$, $\boldsymbol{B}^{f'}$, and $t_{sa}$ are the same as in Theorem 8.*

The proof is rather similar to Lemma 2 but additionally requires the fact that for all worlds $w$ and stochastic action $a$, $\sum_{\{a':a'\sim_w a\}} l(a) = 1$. Unfortunately, the techniques for sensing actions to construct a distribution as Lemma 3 do not apply to stochastic actions. This is because: 1) there is not an explicit inverse of $f$ in terms of $f'$; and 2) the correspondence between $\mathcal{C}$ and $\mathcal{C}'$ breaks. More concretely, assuming $\{a_1, a_2, \dots a_m\}$ are mutual alternatives, given $w' \in \mathcal{C}'$, there might be a set of world $\{w_1, w_2 \dots\}$ s.t. $w' = (w_i)_{a_i}$ and $w_i \in \mathcal{C}_i$ for different $\mathcal{C}_i$. Conversely, given a world $w \in \mathcal{C}$, $w_{a_i}$ might belong to different $\mathcal{C}'_i$.

To solve this problem, we first consider the case where action alternatives are finite.

**Lemma 5** *(Finite action alternatives). Let $\boldsymbol{B}^f$, $\boldsymbol{B}^{f'}$, and $\Sigma$ be as in Theorem 8, $t_{sa}$ a stochastic action with finitely many alternatives under $\Sigma$. For all $d' \in \mathcal{D}$ such that $\{d'\} \models \boldsymbol{B}^{f'} \wedge \boldsymbol{K}\Sigma$, there exists a distribution $d \in \mathcal{D}$ s.t. $\{d\} \models \boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma$ and $d_{t_{sa}} = d'$.*

In the following, we only consider the case that $\text{EQ}(d', \mathcal{W}_{\text{TRUE}}, 1)$. If $\text{EQ}(d', \mathcal{W}_{\text{TRUE}}, c)$ for $c \neq 1$, a distribution can be constructed in the same way except that the weight of worlds is proportionally increased by $c$.

We first observe that given a world $w$ s.t. $w \models \vec{h} = \vec{n} \wedge \Sigma$ for some $\vec{n}$, due to the finite-alternative hypotheses, there are only finitely many alternatives $\{a_{\vec{n},1}, a_{\vec{n},2}, \dots a_{\vec{n},m}\}$ whose likelihoods are positive. Moreover, $w$ might progress to $m$ worlds $w_{a_{\vec{n},1}}, w_{a_{\vec{n},2}}, \dots w_{a_{\vec{n},m}}$ with equivalence class $\mathcal{C}'_{a_{\vec{n},i}}, \mathcal{C}'_{a_{\vec{n},2}} \dots \mathcal{C}'_{a_{\vec{n},m}}$. Conversely, given $m$ equivalence classes $\mathcal{C}'_{a_{\vec{n},1}}, \mathcal{C}'_{a_{\vec{n},2}} \dots \mathcal{C}'_{a_{\vec{n},m}}$, there exists a world $w$ s.t. $w \models \vec{h} = \vec{n} \wedge \Sigma$ and $w_{a_{\vec{n},i}} \in \mathcal{C}'_{a_{\vec{n},i}}$ for all $1 \leq i \leq m$. In fact, there are infinitely many such worlds: the condition $w_{a_{\vec{n},i}} \in \mathcal{C}'_{a_{\vec{n},i}}$ only restricts how $w$ interprets terms $t \in \mathcal{P}_{\vec{\mathcal{H}}}$ after actions $a_{\vec{n},i}$ for $1 \leq i \leq m$, however, worlds might interpret $t$ differently when $z \neq a_{\vec{n},i}$. We only need to select one such world and denote the selected world as $w_{\vec{n},\mathcal{C}'_{a_{\vec{n},1}},\mathcal{C}'_{a_{\vec{n},2}}\dots\mathcal{C}'_{a_{\vec{n},m}}}$. Clearly, we only care about equivalence classes in $\mathcal{W}_{cov}(d')$, namely, $\mathcal{C}'_{a_{\vec{n},i}} \in \mathcal{W}_{cov}(d')$. Intuitively, the selected worlds form the support of $d$, i.e. $S(d)$. Since $m$ is bounded and $\mathcal{C}'_{a_{\vec{n},i}}$ has countably many possible values, $S(d)$ is countable.

Fig. 4 illustrates the selection procedure. Each rectangular box stands for a world and the text over the box indicates the equivalence class it belongs to. The size of the box indicates the relative weight of the world. Boxes on the LHS are worlds that satisfy $\vec{h} = \vec{n}$ and $\Sigma$, their possible progressed worlds are presented on the RHS. The selected world $w_{\vec{n},\mathcal{C}_1}$ corresponds to the combination $\mathcal{C}'_{a_{\vec{n},2}} = \mathcal{C}'_2$ and $\mathcal{C}'_{a_{\vec{n},i}} = \mathcal{C}'_1$ for all $i \neq 2$ (boxes filled by gray), while $w_{\vec{n},\mathcal{C}_2}$ corresponds to the combination $\mathcal{C}'_{a_{\vec{n},2}} = \mathcal{C}'_1$ and $\mathcal{C}'_{a_{\vec{n},i}} = \mathcal{C}'_2$ for all $i \neq 2$ (boxes with dashed boarder). Note that our selection automatically guarantees that different combinations will select different worlds due to the restriction $w_{a_{\vec{n},i}} \in \mathcal{C}'_{a_{\vec{n},i}}$.

Now consider a distribution $d$ as follows:

$$d(w) = \begin{cases} f(\vec{n}) \displaystyle\prod_{i=1}^{m} \frac{d'(w_{a_{\vec{n},i}})}{f'(\vec{n}'_i)} & \begin{array}{l} w = w_{\vec{n},\mathcal{C}'_{a_{\vec{n},1}},\mathcal{C}'_{a_{\vec{n},2}}\dots\mathcal{C}'_{a_{\vec{n},m}}} \text{ for some } \vec{n}, \mathcal{C}'_{a_{\vec{n},1}}, \mathcal{C}'_{a_{\vec{n},2}}, \dots \mathcal{C}'_{a_{\vec{n},m}} \text{ and } w \models \bigwedge_i [a_{\vec{n},i}]\vec{h} = \vec{n}'_i \\ \text{for some } \vec{n}'_1, \vec{n}'_2, \dots \vec{n}'_m \text{ and } d'(w_{a_{\vec{n},i}}) > 0 \text{ for all } i \end{array} \\ 0 & o.w. \end{cases}$$

The construction is based on the observation that for every selected world $w_{\vec{n},\mathcal{C}'_{a_{\vec{n},1}},\mathcal{C}'_{a_{\vec{n},2}}\dots\mathcal{C}'_{a_{\vec{n},m}}}$, the proportion of its weight in $d$ wrt the summed weight of all worlds which satisfy $\vec{h} = \vec{n} \wedge \Sigma$ in $d$, i.e. $f(\vec{n})$, equals the product of proportions of individual progressed world's weight in $d'$ wrt the summed weight in $d'$ of all worlds which assign the same values to fluents in $\mathcal{H}$, i.e. $f'(\vec{n}'_i)$. The distribution $d$ satisfies $\{d\} \models \boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma$ and $d_{t_{sa}} = d'$ (Detailed in Appendix A.9).

While the above construction shows that Lemma 5 indeed holds for the finite alternatives case, it cannot be generalized to the infinite alternatives case. A direct reason is that infinite $\mathcal{C}'_{a_{\vec{n},i}}$ ($i$ is not bounded) need to be considered which results in combinations of infinite dimensions where each dimension has countably infinite candidates. Consequently, $S(d)$ is uncountable. Nevertheless, a weaker lemma exists and is sufficient to prove the progression theorem.

**Lemma 6.** *Let $\boldsymbol{B}^f$, $\boldsymbol{B}^{f'}$, and $\Sigma$ be as in Theorem 8, $t_{sa}$ a stochastic action. For all $d' \in \mathcal{D}$ such that $\{d'\} \models \boldsymbol{B}^{f'} \wedge \boldsymbol{K}\Sigma$ and any $\epsilon > 0$, there exists a distribution $d \in \mathcal{D}$ s.t. $\{d\} \models \boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma$ and $\rho(d_{t_{sa}}, d') < \epsilon$.*

The idea is to only consider a finite subset of $t_{sa}$'s alternatives and construct a distribution $d$ using the above procedure wrt the finite set of alternatives. It can be shown that $d$ satisfies $\boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma$. Additionally, by increasing the size of the finite set of alternatives, the distance $\rho(d_{t_{sen}}, d')$ decreases accordingly and is eventually less than $\epsilon$ (Detailed in Appendix A.10).

**Proof of Theorem 8.** Suppose $e \models \boldsymbol{O}(\boldsymbol{B}^f \wedge \Sigma)$ and $e' \models \boldsymbol{O}(\boldsymbol{B}^{f'} \wedge \Sigma)$, by Theorem 4, it suffices to show $e_{t_{sen}} = e'$

"$e_{t_{sen}} \subseteq e'$": the proof is exactly the same as its counterpart for noisy sensing.

"$e' \subseteq e_{t_{sen}}$": Since $e' \models \boldsymbol{O}(\boldsymbol{B}^{f'} \wedge \boldsymbol{K}\Sigma)$, for all $d' \in e'$, $\{d'\} \models \boldsymbol{B}^{f'} \wedge \boldsymbol{K}\Sigma$. By Lemma 6, $\exists d.\{d\} \models \boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma$ and $\rho(d_{t_{sen}}, d') < \epsilon$ for any $\epsilon$. Therefore, $\exists\{d_1, d_2, \ldots d_i \ldots\}$ s.t. $\{d_i\} \models \boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma$ and there exists $N$ for $i > N$, $\rho((d_i)_{t_{sen}}, d') < \epsilon$ for any $\epsilon$. That is, $\exists\{d_1, d_2, \ldots d_i \ldots\}$ s.t. $\{d_i\} \models \boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma$ and $\lim_{i\to\infty}(d_i)_{t_{sa}} = d'$. By definition of $e_{t_{sen}}$, $d' \in e_{t_{sen}}$. $\square$

**Example 4.** Let $\boldsymbol{B}^f$ and $\Sigma$ be as Example 2, then $\boldsymbol{O}(\boldsymbol{B}^f \wedge \Sigma) \models [fwd(2,2)]\boldsymbol{O}(\boldsymbol{B}^{f'} \wedge \Sigma)$ where $f'$ is given by

$$f'(u) = \begin{cases} 1/15 & u = 2 \\ 4/15 & u = 1 \\ 2/3 & u = 0 \\ 0 & o.w. \end{cases}$$

By definition, we have:

$$\mathbb{I}(u, u', a, fwd(2,2)) = \begin{cases} 1 & Pro(h = u', a)^h_u \wedge \psi^{a'}_{fwd(2,2)} \\ 0 & o.w. \end{cases}$$

$$= \begin{cases} 1 & \begin{array}{c} (\exists x, y.a = fwd(x,y) \\ \wedge u = \max\{0, u' - y\} \vee \forall x, y.a \neq fwf(x,y) \\ \wedge u = u') \wedge \exists y.a = fwd(2,y) \end{array} \\ 0 & o.w. \end{cases} = \begin{cases} 1 & \exists y.a = fwd(2,y) \wedge u = \max\{0, u' - y\} \\ 0 & o.w. \end{cases}$$

and $\mathcal{L}(a)^h_{u'} = \begin{cases} \theta(x, y, 0.2, 0.6) & \exists x, y.a = fwd(x,y) \\ \theta(z, u', 0.1, 0.8) & \exists z.a = sonar(z) \end{cases}$, therefore

$$f'(u) = \sum_{u'} \sum_a f(u')\mathcal{L}(a)^h_{u'}\mathbb{I}(u, u', a, fwd(2,2))$$

$$= \sum_{u'} \sum_a f(u') \begin{cases} \theta(2, y, 0.2, 0.6) & \begin{array}{c} \exists y.a = fwd(2,y) \wedge \\ u = \max\{0, u' - y\} \end{array} \\ 0 & o.w. \end{cases} = \begin{cases} \frac{1}{3} \times 0.2 & u = 2 \\ \frac{1}{3} \times 0.2 + \frac{1}{3} \times 0.6 & u = 1 \\ \begin{pmatrix} \frac{1}{3} \times 0.2 + \frac{1}{3} \times 0.6 + \frac{1}{3} \times 0.2 \\ +\frac{1}{3} \times 0.2 + \frac{1}{3} \times 0.6 + \frac{1}{3} \times 0.2 \end{pmatrix} & u = 0 \\ 0 & o.w. \end{cases}$$

The second line is by $\mathcal{L}(a)^h_u \times \mathbb{I}(u, u', a, fwd(2,2))$. The third is because $f$ and $\theta$ are zero when $u'$ and $y$ are not among $\{1, 2, 3\}$, respectively, therefore $f'$ is non-zero only if $u \in \{0, 1, 2\}$ and the degree of belief for each value of $u$ equals to the sum of products between $f(u')$ and $\theta(2, y, 0.2, 0.6)$ of all combinations of $u'$ and $y$ which result that value (according to $u = \max\{0, u' - y\}$).

## 4. Related work

To the best of our knowledge, this work is the first work to study the relationship between the notion of only-believing and the topological structure of distributions entertained by the epistemic state. Below, we revisit related work from two aspects: knowledge representation languages and projection by progression.

In terms of knowledge representation languages, our logic is built on the logic $\mathcal{DS}$, a probabilistic extension of a modal variant of the situation calculus with a model of belief and only-believing. What distinguishes us is that our proposed logic has richer expressiveness that allows us to express a probabilistic knowledge base with arbitrary belief distributions and the utility of our notion of only-believing is not restricted to the static case. The technique of special treatment on rigid terms, which further enables us to specify arbitrary belief distributions, is drawn from [13]. The logic $\mathcal{DS}$ is based on the first-order logic $\mathcal{OBL}$ [6], a modal probabilistic logic of only-knowing. While $\mathcal{OBL}$ considers beliefs and only-believing in static cases, $\mathcal{DS}$ incorporates actions as well. It is shown that $\mathcal{OBL}$ fully captures the features of the logic $\mathcal{OL}$ (see the pioneering work on only-knowing by [21]). In a game theory context, Halpern and Pass have considered a (propositional) version of only-knowing with probability distributions [22]. The notion of only-knowing in the above works is only from a single agent's perspective. Multi-agent only-knowing is studied in [23–25]. It is recognized that only-knowing has a close relationship with non-monotonic reasoning [26] including the autoepistemic logic by Moore [27,28] and Reiter's default logic [29]. See [30] for discussion.

For the aspect of degrees of belief, $\mathcal{DS}_p$ is inspired by the work BHL [3], an axiomatic proposal with a conceptually attractive definition of belief in a first-order dynamic setting. Later, the account is extended to express continuous distribution [31]. In a less restricted setting, reasoning about knowledge and probability was studied before BHL [32–34]. Notably, the work of Fagin and Halpern [33] can be seen to be at the heart of BHL, see [12] for a comprehensive discussion on related works in line of research. Limited versions of probabilistic logic like graphical models or similar can be found in [35–42]. Specifically, based on a logic that encodes a Bayesian net, [35] proposed an algorithm to infer propositional probabilistic knowledge. [37] studied inference in FO Bayesian net. [38] investigated a tractable subset of FO probabilistic logic, the Markov logic [41]. Probabilistic program [39] and probabilistic database [40] are also directions to unify logic and probability. Roughly speaking, these proposals sacrifice the expressiveness of a more powerful language to obtain decidability or tractability in inference [35] and learnability [42,36].

Regarding progression, Lin and Reiter proposed the most general account of progression and showed that progression is only second-order definable [7]. The notion of logical filtering, which is a kind of progression, is studied in [43,44]. Restricted forms of the LR-progression, which are first-order definable, are discussed in [7,18,45,46]. Based on the notion of progressed worlds, Lakemeyer and Levesque show that the progression of categorical knowledge against noise-free actions amounts to only-knowing after actions [8]. In a stochastic setting, for a limited type of theory, the progression of discrete degrees of belief wrt context-completeness is considered in [47]. The progression of continuous degrees of belief for the so-called *invertible* BATs, which exclude our BATs in Example 2, is studied in [19]. As a result, our work fills the gap of a general account of progression in discrete degrees of belief.

Another related venue of research is *belief change* (or *revision*), which studies how to incorporate new (potentially contradictory) information in a knowledge base. Our logic, stemming from the BHL formalism, has a mechanism to incorporate consistent information: beliefs change in the way of Bayesian update after consistent sensing, nevertheless, when sensing an unexpected result, the agent's epistemic state becomes inconsistent and it believes everything. Belief revision aims at revising the epistemic state so that the agent still has reasonable beliefs when inconsistent information is obtained. The initial proposal on belief revision is by Alchourrón, Gärdenfors and Makinson [48,49] (AGM), where eight postulates were proposed to evaluate a revision function. Many efforts have been made in extending the situation calculus to address belief change [50–54]. Notably, [51,52] focused on belief revision with sensing, fallible actions, and nondeterministic actions. [53] used plausibility rankings on worlds and actions to model epistemic states that change according to actions, later [54] studied progression under such a framework but only considered the propositional fragment. Belief revision for conditional belief was studied in [55]. Nevertheless, none of the above works considered probability except [56]. It is unclear how ideas can be used in the logic $\mathcal{DS}_p$ to model belief revision, perhaps, instead of maintaining a set of distributions as the epistemic state, one needs to use a ranking function on such sets just like the classical plausibility ranking in [51–53].

## 5. Discussion on the closure operator in $e_z$

Before concluding, we want to take a closer look at the relationship between the notion of only-believing and the topological structure of distributions entertained by the epistemic state. There are several interesting questions.

To begin with, why do we enforce $e_z$ to be a closed set and what are the consequences? Previously, the first question is answered by Theorem 1, namely, the set $\{d|\textsc{Norm}(d, \mathcal{W}_\alpha^{\{d\},\lambda}, \mathcal{W}_{\textsc{true}}, |r|_\lambda)\}$ is closed for any objective $\alpha, \lambda$, and $r$. However, what if $\alpha$ is subjective? In fact, Theorem 1 breaks if $\alpha$ is subjective.

**Proposition 5.** *For any $\alpha, \lambda$, the set $e = \{d|\textsc{Norm}(d, \mathcal{W}_{\neg\boldsymbol{K}\alpha}^{\{d\},\lambda}, \mathcal{W}_{\textsc{true}}, 1)\}$ is not closed.*

**Proof.** This is not hard to see. $\textsc{Norm}(d, \mathcal{W}_{\neg\boldsymbol{K}\alpha}^{\{d\},\lambda}, \mathcal{W}_{\textsc{true}}, 1)$ iff $\mathcal{W}_{\neg\boldsymbol{K}\alpha}^{\{d\},\lambda} = \mathcal{W}_{\textsc{true}}$ iff (by Def. of $\mathcal{W}_{\neg\boldsymbol{K}\alpha}^{\{d\},\lambda}$) $\forall w \in \mathcal{W}, \{d\}, w \models \neg\boldsymbol{K}\alpha$ iff $\neg\textsc{Norm}(d, \mathcal{W}_\alpha^{\{d\},\lambda}, \mathcal{W}_{\textsc{true}}, 1)$. Hence $e = \{d|\neg\textsc{Norm}(d, \mathcal{W}_\alpha^{\{d\},\lambda}, \mathcal{W}_{\textsc{true}}, 1)\}$.

Intuitively, $e$ contains all the distributions which believe $\alpha$ with a degree less than 1. Therefore, the set of distributions believing $\alpha$ with degree 1 ($\boldsymbol{K}\alpha$) will be excluded even if they are limit points of some sequence of distribution in $e$. More concretely, let $\{d_1, d_2, \ldots d_i\}$ be a sequence of distribution s.t. $d_i \in \mathcal{D}$ and $\textsc{Norm}(d_i, \mathcal{W}_\alpha^{\{d_i\},\lambda}, \mathcal{W}_{\textsc{true}}, |1 - \frac{1}{i+1}|)$, it is easy to see for all $i, d_i \in e$ and for the limit $\hat{d} = \lim_{i\to\infty} d_i$, we have $\textsc{Norm}(d_i, \mathcal{W}_\alpha^{\{d_i\},\lambda}, \mathcal{W}_{\textsc{true}}, 1)$, hence $\hat{d} \notin e$. $\square$

As a direct consequence, we have:

**Proposition 6.** *For any sentence $\alpha$, $\boldsymbol{O}(\neg\boldsymbol{K}\alpha)$ is unsatisfiable in $\mathcal{DS}_p$.*

However, the formula $\boldsymbol{O}(\neg\boldsymbol{K}\alpha)$ makes perfect sense: intuitively, the agent only believes $\alpha$ with some degree of belief that is less than 1.

A comment is that, for the same reason that $e_z$ is enforced to be closed, Theorem 2 only holds for strictly positive beliefs. The reason behind this is that the intersection of closed sets is also closed, however, the complement and union of closed sets are not necessarily closed.

Proposition 6 roughly speaking answers the second question. With this undesirable property, why do we enforce $e_z$ to be closed at all? The answer is: if $e_z$ is not closed, the progression theorem (Theorem 8) will no longer hold.

To distinguish the semantics of $\boldsymbol{O}$ (where $e_z$ is enforced to be closed), we introduce a new modality $\boldsymbol{O}''$ and truth of $\boldsymbol{O}''$ is given as:

- $e, w, z, \lambda \models \boldsymbol{O}''(\alpha : r)$ iff
  $\forall d', d' \in \{d_z | d \in e \cap \mathcal{D}, d \sim_{comp} z\}$ iff $\text{NORM}(d', \mathcal{W}_\alpha^{\{d'\},\lambda}, \mathcal{W}_{\text{TRUE}}^{\{d'\},\lambda}, n)$ for $n \in \mathbb{R}$, and $n = |r|_\lambda$.

Namely, $\boldsymbol{O}''$ is the version of only-believing where $e_z$ is not enforced to be closed. By the semantics of $\boldsymbol{O}''$, for any $\alpha, r$, we have $e \models \boldsymbol{O}''(\alpha : r)$ iff $e = \{d | \text{NORM}(d, \mathcal{W}_\alpha, \mathcal{W}_{\text{TRUE}}, |r|)\}$. Additionally, one can prove that $\boldsymbol{O}''(\neg \boldsymbol{K}\alpha)$ is satisfiable now. Nevertheless, we also have

**Theorem 9.** *There exists a KB $\boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma$ and a stochastic action $t_{sa}$, $\boldsymbol{O}''(\boldsymbol{B}^f \wedge \Sigma) \not\models [t_{sa}]\boldsymbol{O}''(\boldsymbol{B}^{f'} \wedge \Sigma)$, where $f'$ is a definitional function given as Theorem 8.*

**Proof.** Consider our robot moving domain again where a fluent $h$ indicates the robot's distance from the wall. Now, assuming that the robot can only perform a stochastic action $set(x)$ to move. Formally, we have $\Sigma$ as[10]:

$\Box[a]h = u \equiv \exists x. a = set(x) \wedge u = x \ \vee \forall x. a \neq set(x) \wedge h = u$

$\Box oi(a, a') = \top \equiv \exists x, x'. a = set(x) \wedge a' = set(x')$

$\Box l(a) = \mathcal{L}(a)$ with $\mathcal{L}(a) = \begin{cases} 2^{-i} & \exists x. a = set(x) \wedge \exists i \in \mathbb{N}^+. x = \frac{1}{i} \\ 0 & o.w. \end{cases}$

Intuitively, the BAT says that the distance $h$ can be set to values in $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ by the $set(x)$ action and the likelihood of $set(x)$ is independent of the current distance. Let $\boldsymbol{B}^f$ be the belief distribution as $f(u) = 1$ if $u = 1$ and 0 otherwise. According to Lemma 4, we have $\boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma \models \boldsymbol{B}^{f'} \wedge \boldsymbol{K}\Sigma$ where $f'$ is given by

$$f'(u) = \begin{cases} 2^{-i} & \exists i \in \mathbb{N}^+. u = \frac{1}{i} \\ 0 & o.w. \end{cases}.$$

Namely, $\boldsymbol{B}^{f'}$ is exactly the same as the distribution of the action $set(x)$'s likelihood. This is trivial as no matter what is believed initially, after the action $set(x)$, the belief distribution will be set to the likelihood distribution of $set(x)$.

Let $e \models \boldsymbol{O}''(\boldsymbol{B}^f \wedge \Sigma)$ and $e' \models \boldsymbol{O}''(\boldsymbol{B}^{f'} \wedge \Sigma)$, it suffices to show that $e_{t_{sa}} \neq e'$. Now, consider a set of worlds $\{w_{1,1}, w_{2,1}, w_{2,2}, w_{3,1} \ldots\}$ (for index $i$, there are $i$ worlds $w_{i,1}, w_{i,2}, \ldots w_{i,i}$) s.t. for distinct $j, j'$, $w_{i,j} \neq w_{i,j'}$ and $w_{i,j} \models h = \frac{1}{i} \wedge \Sigma$. Now we define $d'$ as $d'(w_{i,j}) = \frac{1}{i} \times f'(\frac{1}{i}) = \frac{1}{i} \times 2^{-i}$ for all $i, j$, that is, we assign the weight $f'(\frac{1}{i})$ equally to worlds $w_{i,1}, w_{i,2}, \ldots w_{i,i}$. Fig. 5 demonstrates the construction of $d'$, where the length of boxes on the RHS of the arrows denotes the weight of worlds in $d'$. Clearly $\{d'\} \models \boldsymbol{B}^{f'} \wedge \boldsymbol{K}\Sigma$, therefore $d' \in e'$. Now we prove $d' \notin e_{set(0)}$, namely, $\neg \exists d \in e$ s.t. $d_{set(0)} = d'$.

Assuming that $\exists d \in e$ s.t. $d_{set(0)} = d'$, since $d \in e$, $\{d\} \models \boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma$. For any $w \in S(d)$, clearly $\exists j \in \mathbb{N}^+$ s.t. $\frac{1}{2}d(w) > \frac{1}{j}$. For the world $w_{set(\frac{1}{j})}$, since $d_{set(0)} = d'$ by assumption, $d'(w_{set(\frac{1}{j})}) > 0$ by definition of $d_z$ and our construction of $\Sigma$. Hence, $w_{set(\frac{1}{j})} \in S(d')$. On the other hand, $w_{set(\frac{1}{j})} \models h = \frac{1}{j} \wedge \Sigma$, therefore $d'(w_{set(\frac{1}{j})}) = \frac{1}{j} f'(\frac{1}{j})$. As a result:

$$\frac{1}{2}d(w) > \frac{1}{j} > \frac{d'(w_{set(\frac{1}{j})})}{f'(\frac{1}{j})} = \frac{1}{2^{-j}} \sum_{\{w' | d(w') > 0\}} \sum_{\{a | a \sim_{set(\frac{1}{j})}, (w')_a = w_{set(\frac{1}{j})}\}} d(w') l^*(w', a)$$

$$> \frac{1}{2^{-j}} d(w) l^*(w, set(\frac{1}{j})) = d(w)$$

Contradiction. Therefore the assumption does not hold. The second line is by definition of $d_z$ while the third line is by only considering the single world $w$ in $S(d)$. $\square$

Another interesting question is whether we need to impose the closure operator $cl(\cdot)$ *explicitly*, i.e. whether the set of distributions $\{d_z | d \in e \cap \mathcal{D}, d \sim_{comp} z\}$ by Definition 7 is already closed. The answer is no, i.e. the set $\{d_z | d \in e \cap \mathcal{D}, d \sim_{comp} z\}$ is not always closed, therefore $cl(\cdot)$ is required explicitly.

[10] Since natural numbers are used, implicitly, we add the theory of natural number as premise for logical consequences.

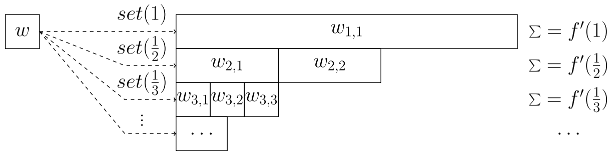


**Fig. 5.** The construction of $d'$.

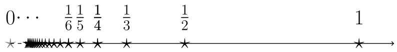


**Fig. 6.** $h$'s possible values $\mathbb{H}$ in Example 5.

**Example 5.** Consider our robot moving domain again where a fluent $h$ is used to indicate the robot's distance from the wall. The robot can perform a stochastic action $set(x)$ to move, where the outcome $x$ determines its new location. Formally, we have:

$\Box[a]h = u \equiv \exists x.a = set(x) \wedge u = x \ \vee \forall x.a \neq set(x) \wedge h = u$

$\Box oi(a, a') = \top \equiv \exists x, x'.a = set(x) \wedge a' = set(x')$

$$\Box l(a) = \mathcal{L}(a) \text{ with } \mathcal{L}(a) = \begin{cases} 2^{-h} & a = set(0) \\ 1 - 2^{-h} & \exists x.a = set(x) \wedge x = h \\ 0 & o.w. \end{cases}$$

See Fig. 6 for $h$'s possible values. The BAT is rather similar to the one in the proof of Theorem 9 except that now we assume the likelihood of $set(x)$ depends on the robot's location: after the $set(x)$ action, the robot's location might turn to 0 with a positive likelihood $2^{-h}$ due to the outcome $set(0)$ and remain unchanged with likelihood $1 - 2^{-h}$ for the outcome $set(x)$ where $x$ is the current location.

Let $\mathbb{H}^-$ be the infinite set $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ and $\mathbb{H} = \mathbb{H}^- \cup \{0\}$. Now consider a countably infinite set of worlds $\{w_0, w_1 \ldots\}$ s.t. 1) $w_0 \models h = 0 \wedge \Sigma$; 2) for $i \geq 1$, $w_i \models h = \frac{1}{i} \wedge \Sigma$ and $(w_i)_{set(0)} = w_0$. Trivially, $(w_i)_{set(x)} \neq w_0$ for $x \neq 0$ by $\Sigma_{post}$ above.

For the epistemic state $e = \{d | \sum_i d(w_i) = 1$, and $\forall w \notin \{w_1, w_2 \ldots\}, d(w) = 0\}$, we have:

1. $e$ is closed;
2. $\forall d' \in \{d_{set(0)} | d \in e \cap \mathcal{D}, d \sim_{comp} set(0)\}$, $d'(w_0) < 1$.

The first item is trivially by definition of closure. As for the second, we have

$$\begin{aligned} d'(w_0) = d_{set(0)}(w_0) &= \sum_{\{w|d(w)>0\}} \sum_{\{a|a \sim_w set(0), w_a = w_0\}} d(w) \times l^*(w, a) \\ &= \sum_{i=1}^{\infty} \sum_{\{x|(w_i)_{set(x)} = w_0\}} d(w_i) \times l^*(w_i, set(x)) \\ &= \sum_{i=1}^{\infty} d(w_i) \times l^*(w_i, set(0)) \\ &= \sum_{i=1}^{\infty} d(w_i) \times 2^{-\frac{1}{i}} < \sum_{i=1}^{\infty} d(w_i) = 1 \end{aligned}$$

The first line is by definition of $d_z$, while the second line is by our construction of $e$. The third line is because $(w_i)_{set(x)} = w_0$ only if $x = 0$. The last line is simply by the likelihood of $set(0)$.

Intuitively, $e$ is the set of all distributions which says that $h$'s possible values are among $\mathbb{H}^-$, therefore $e$ is closed. Additionally, for such distributions, after the action $set(0)$, they would not shift all the probability mass to the point $h = 0$ ($d'(w_0) < 1$) since with some positive likelihood, $h$ remains unchanged.

Now consider the sequence of distribution $\{d_1, d_2, \ldots d_j, \ldots\}$ where $d_j(w) = 1$ if $w = w_j$ else $d_j(w) = 0$. Trivially, we have 1) $d_j \in e$; 2) $(d_j)_{set(0)}(w_0) = 1 \times l^*(w_j, set(0)) = 2^{-\frac{1}{j}}$. Therefore for $\hat{d} = \lim_{j\to\infty}(d_j)_{set(0)}$, $\hat{d}(w_0) = \lim_{j\to\infty}(d_j)_{set(0)}(w_0) = \lim_{j\to\infty} 2^{-\frac{1}{j}} = 1$. Hence $\hat{d} \notin \{d_{set(0)} | d \in e \cap \mathcal{D}, d \sim_{comp} set(0)\}$. $\{d_{set(0)} | d \in e \cap \mathcal{D}, d \sim_{comp} set(0)\}$ is not closed.

The above example suggests that the $cl(\cdot)$ is *explicitly* required in the definition of $e_z$.

## 6. Conclusion

Rich representations of knowledge and actions have been a goal that many AI researchers have pursued. Efforts include formalisms such as dynamic logic [57], the action language [58], and the fluent calculus [59]. Our formalism essentially derives from the situation calculus by [16].

In this work, we

- lift the expressiveness of the logic $\mathcal{DS}$ by our special treatment on rigid terms. As a result, it is possible to express belief distributions as well as only-believing with a belief distribution in our logic;
- study the notion of the progressed epistemic state which might entertain multiple possible distributions over possible worlds. More concretely, the progressed epistemic state turns out to be, in many cases, a closed set on the metric space formed by the set of all discrete distributions;
- study the progression of a type of probabilistic knowledge base specified by only-believing with a belief distribution. For both stochastic actions and sensing actions, it turns out that the progression can be defined by only-believing with another belief distribution after that action, which generalizes the result in [8].

In terms of future work, there are several questions arising from this work. Firstly, the discussion in Section 5 suggests a dilemma of whether to use the closure operator in $e_z$. A question is if there are other ways to define the notion of only-believing so that such a dilemma can be avoided. Our sense of the answer to this problem is negative, as long as the maximal semantics is used and the progressed distribution is updated in a Bayesian way as in our logic. This is because the example in the proof of Theorem 9 suggests that, in some domains, some discrete distributions cannot be progressed through discrete distributions. This also stirs up another interesting problem: will the dilemma persist when considering continuous domains?

Secondly, as discussed before, the set of computable numbers $\mathbb{C}$ we use is not closed under summation operators [14]. The trivial solution provided in this work reserves a special standard name *undefined* for summations whose values are not in the domain. A consequence is that possibly two expressions with summations that are not equal in the usual sense are logically equivalent in the logic $\mathcal{DS}_p$. Another possible solution that might overcome this is to enlarge the domain. For example, instead of using the computable numbers $\mathbb{C}$, one could consider the *definable number* $\mathbb{D}$ [60]. However, it is unclear whether $\mathbb{D}$ is closed under summation.

Another direction of research is to study how the proposed probabilistic formalism can be adapted to the context of belief revision [49,61]. Belief revision, as discussed before, is the process of changing beliefs to take into account a new piece of information. For example, in the event such as the agent's actions fail or sensing returns an unexpected value, a rational agent should revise her belief accordingly. In the logic $\mathcal{DS}_p$, once the sensing result contradicts the KB, the agent believes everything as $e$ turns to empty which is undesirable in some scenarios. Shapiro, Pagnucco, Lesperance and Levesque provide an account for reasoning about action and belief change in the situation calculus by the notion of plausibility [50]. It remains interesting to see how these ideas can be incorporated within $\mathcal{DS}_p$.

On the application side, while leveraging the logic $\mathcal{DS}_p$ to do epistemic planning under uncertainty is definitely a promising direction, it is also interesting to see how the proposed formalism can be used in the verification of belief programs [62].

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

### Data availability

No data was used for the research described in the article.

### Acknowledgements

This work has been supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) RTG 2236 'UnRAVeL' and by the EU ICT-48 2020 project TAILOR (No. 952215). Special thanks to Gerhard Lakemeyer for his invaluable comments.

## Appendix A. Long proofs

### A.1. Proof of Proposition 1

**Proof of Proposition 1.** Only need to show $\rho$ satisfies the properties of a metric. Namely, for all $d, d', d''$, 1) $\rho(d, d') = 0$ iff $d = d'$ (identity of indiscernibles); 2) $\rho(d, d') = \rho(d', d)$ (symmetry); 3) $\rho(d, d') + \rho(d', d'') \geq \rho(d, d'')$ (triangle inequality).

The identity of indiscernibles and symmetry properties are trivial to prove, while for triangle inequality, we have:

$$\begin{aligned}\rho(d,d')+\rho(d',d'') &= \sum_{w\in S(d)\cup S(d')} |d(w)-d'(w)| + \sum_{w\in S(d')\cup S(d'')} |d'(w)-d''(w)| \\ &= \sum_{w\in S(d)\cup S(d')\cup S(d'')} |d(w)-d'(w)| + \sum_{w\in S(d)\cup S(d')\cup S(d'')} |d'(w)-d''(w)| \geq \sum_{w\in S(d)\cup S(d')\cup S(d'')} |d(w)-d''(w)| \\ &= \sum_{w\in S(d)\cup S(d'')} |d(w)-d''(w)| = \rho(d,d'')\end{aligned}$$

The second equality is due to that $d(w)=0$ for $w \notin S(d)$. Hence $|d(w)-d'(w)|=0$ for $w \in S(d'')\backslash S(d)\cup S(d')$, likewise $|d'(w)-d''(w)|=0$ for $w\in S(d)\backslash S(d')\cup S(d'')$. □

### A.2. Proof of Theorem 1

**Proof of Theorem 1.** When $e = cl(e)$, the theorem holds trivially. For $e \neq cl(e)$, suppose $\exists d', s.t. d' \in cl(e)\backslash e$, then $d' = \lim_{i\to\infty} d_i$ for some $\{d_1, d_2, \ldots d_i, \ldots\}$ where $d_i \in e$, we show $\textsc{Norm}(d', \mathcal{W}_\alpha^{\{d'\},\lambda}, \mathcal{W}_{\text{TRUE}}, |r|_\lambda)$. Let $S' = (\bigcup_{i\in\mathbb{N}} S(d_i)) \cup S(d')$, clearly $S'$ is still a countable set.

$$\begin{aligned}\frac{\sum_{\{w:w\in S(d'),w\models\alpha\}} d'(w)}{\sum_{w\in S(d')} d'(w)} &= \frac{\sum_{\{w:w\in S(d'),w\models\alpha\}} \lim_{i\to\infty} d_i(w)}{\sum_{w\in S(d')} \lim_{i\to\infty} d_i(w)} && \text{(Prop. 2)} \\ &= \frac{\sum_{\{w:w\in S',w\models\alpha\}} \lim_{i\to\infty} d_i(w)}{\sum_{w\in S'} \lim_{i\to\infty} d_i(w)} \\ &= \frac{\lim_{i\to\infty} \sum_{\{w:w\in S',w\models\alpha\}} d_i(w)}{\lim_{i\to\infty} \sum_{w\in S'} d_i(w)} && \text{(absolute convergence)} \\ &= \frac{\lim_{i\to\infty} \sum_{\{w:w\in S(d_i),w\models\alpha\}} d_i(w)}{\lim_{i\to\infty} \sum_{w\in S(d_i)} d_i(w)} \\ &= \lim_{i\to\infty} \frac{\sum_{\{w:w\in S(d_i),w\models\alpha\}} d_i(w)}{\sum_{w\in S(d_i)} d_i(w)} && \left(\textstyle\sum_{w\in S(d_i)} d_i(w) > 0\right) \\ &= \lim_{i\to\infty} |r|_\lambda = |r|_\lambda && (d_i \in e \text{ and } \textsc{Norm}(d_i, \mathcal{W}_\alpha^{\{d_i\},\lambda}, \mathcal{W}_{\text{TRUE}}, |r|_\lambda))\end{aligned}$$

The second and fourth equality are again due to $d_i(w)=0$ for $w \notin S(d_i)$. □

### A.3. Proof of Theorem 2

The proof relies on a result in set theory: any (countably many) intersections of closed sets are also closed. Namely. if $e_i$ are closed, then $\bigcap_i e_i$ is also closed. This can be proved as follows:

For any sequence $\{d_1, d_2, \ldots, d_j, \ldots\}$ of $\bigcap_i e_i$, supposing $d = \lim_{j\to\infty} d_j$, it suffices to show $d \in \bigcap_i e_i$, namely, $d \in e_i$ for all $i$. Since $d_j \in \bigcap_i e_i$ and $\bigcap_i e_i \subseteq e_i$, $d_j \in e_i$. By the hypothesis that $e_i$ is closed, we have $d \in e_i$.

**Proof of Theorem 2.** In the following, assume $\alpha, \alpha_1, \alpha_2$ are strictly positive beliefs. By the semantics of $\boldsymbol{O}$, $e \models \boldsymbol{O}\alpha$, iff $cl(e) = \{d\colon \textsc{Norm}(d, \mathcal{W}_\alpha^d, \mathcal{W}_{\text{TRUE}}^d, 1)\}$. It suffices to prove that $\{d\colon \textsc{Norm}(d, \mathcal{W}_\alpha^d, \mathcal{W}_{\text{TRUE}}^d, 1)\}$ is a closed set.

The base case is a direct consequence of Theorem 1. We focus on the induction here.

- for $\boldsymbol{K}\alpha$, and $d \in \mathcal{D}$, $\textsc{Norm}(d, \mathcal{W}_{\boldsymbol{K}\alpha}^d, \mathcal{W}_{\text{TRUE}}^d, 1)$ iff $\textsc{Norm}(d, \mathcal{W}_\alpha^d, \mathcal{W}_{\text{TRUE}}^d, 1)$. That is $\{d\colon \textsc{Norm}(d, \mathcal{W}_\alpha^d, \mathcal{W}_{\text{TRUE}}^d, 1)\} = \{d\colon \textsc{Norm}(d, \mathcal{W}_{\boldsymbol{K}\alpha}^d, \mathcal{W}_{\text{TRUE}}^d, 1)\}$. By the induction hypothesis, $\{d\colon \textsc{Norm}(d, \mathcal{W}_{\boldsymbol{K}\alpha}^d, \mathcal{W}_{\text{TRUE}}^d, 1)\}$ is closed;
- for $\alpha_1 \wedge \alpha_2$ (likewise for $\alpha_1 \vee \alpha_2$), and $d \in \mathcal{D}$, $\{d\colon \textsc{Norm}(d, \mathcal{W}_{\alpha_1\wedge\alpha_2}^d, \mathcal{W}_{\text{TRUE}}^d, 1)\} = \{d\colon \textsc{Norm}(d, \mathcal{W}_{\alpha_1}^d, \mathcal{W}_{\text{TRUE}}^d, 1)\} \cap \{d\colon \textsc{Norm}(d, \mathcal{W}_{\alpha_2}^d, \mathcal{W}_{\text{TRUE}}^d, 1)\}$ (this is because $\alpha_1$ and $\alpha_2$ are subjective). By the induction hypothesis, $\{d\colon \textsc{Norm}(d, \mathcal{W}_{\alpha_1}^d, \mathcal{W}_{\text{TRUE}}^d, 1)\}$ and $\{d\colon \textsc{Norm}(d, \mathcal{W}_{\alpha_2}^d, \mathcal{W}_{\text{TRUE}}^d, 1)\}$ are closed, hence their intersection is closed as well;
- for $\forall v.\alpha$, the result of the above applies here as well since quantification is understood substantially and the countable intersection of closed sets is also a closed set.

This completes the proof. □

### A.4. Proof of Proposition 3

**Proof of Proposition 3.**

1. Suppose $e \models \boldsymbol{O}(\alpha_1 : r_1; \ldots \alpha_k : r_k)$, by semantics, $e = \{d : \bigwedge \textsc{Norm}(\mathcal{W}^{\{d\}}_{\alpha_i}, \mathcal{W}^{\{d\}}_{\text{TRUE}}, r_i)\}$, therefore, for all $i$, $e \subseteq \{d : \textsc{Norm}(\mathcal{W}^{\{d\}}_{\alpha_i}, \mathcal{W}^{\{d\}}_{\text{TRUE}}, r_i)\}$, hence $e \models \bigwedge \boldsymbol{B}(\alpha_i : r_i)$.
2. In case $|r'| > 0$, let $w_1, w_2$ be two worlds s.t. $w_1 \models \alpha \wedge \neg h(\vec{n}) = m$ and $w_2 \models \alpha \wedge \neg h(\vec{n}) = m$. Now, consider a distribution $d$: $d(w_1) = |r|$, $d(w_2) = 1 - |r|$, and $d(w) = 0$ for other worlds $w$. Clearly, $\textsc{Norm}(d, \mathcal{W}^{\{d\}}_{\alpha}, \mathcal{W}^{\{d\}}_{\text{TRUE}}, |r|)$ and $\textsc{Norm}(d, \mathcal{W}^{\{d\}}_{h(\vec{n})=m}, \mathcal{W}^{\{d\}}_{\text{TRUE}}, 0)$. For $e \subseteq \mathcal{D}$ s.t. $e \models \boldsymbol{O}(\alpha : r)$, by semantics, $d \in cl(e)$, thus $e \not\models \boldsymbol{B}(h(\vec{n}) = m : r')$.
   In case $r' = 0$, let $w_1 \models \alpha \wedge h(\vec{n}) = m$ and $w_2 \models \alpha \wedge h(\vec{n}) = m$ and $d, e$ be the same as above, likewise, we have $e \not\models \boldsymbol{B}(h(\vec{n}) = m : r')$.
3. This is a direct result of Theorem 1. $\square$

### A.5. Proof of Lemma 1

We begin with a lemma about the distance function $\rho$.

**Lemma 7.** *Given distributions $d, d' \in \mathcal{D}$, for all action $a \in \mathcal{N}_A$ s.t. $d \sim_{comp} a$ and $d' \sim_{comp} a$, $\rho(d_a, d'_a) \leq \rho(d, d')$.*

The proof relies on two facts about $l$, i.e., $w[l(a), z] \leq 1$ and $\sum_{\{a' : a' \sim_w a\}} w[l(a), z] = 1$, for all $w \in \mathcal{W}, z \in \mathcal{Z}$, and $a \in \mathcal{N}_A$.

**Proof.**

$$
\begin{aligned}
\rho(d_a, d'_a) &= \sum_{w \in S(d_a) \cup S(d'_a)} |d_a(w) - d'_a(w)| \\
&= \sum_{w \in S(d_a) \cup S(d'_a)} \Big| \sum_{\{(w', a') : w' \in S(d) \cup S(d'), a' \sim_{w'} a, w'_{a'} = w\}} (d(w') - d'(w')) \times l^*(w', a') \Big| \\
&\leq \sum_{w \in S(d_a) \cup S(d'_a)} \sum_{\{(w', a') : w' \in S(d) \cup S(d'), a' \sim_{w'} a, w'_{a'} = w\}} |d(w') - d'(w')| \times l^*(w', a') \\
&= \sum_{\{(w', a') : w' \in S(d) \cup S(d'), a' \sim_{w'} a\}} |d(w') - d'(w')| \times l^*(w', a') \\
&= \sum_{w' \in S(d) \cup S(d')} |d(w') - d'(w')| \times \sum_{\{a' : a' \sim_{w'} a\}} l^*(w', a') \\
&\leq \sum_{w' \in S(d) \cup S(d')} |d(w') - d'(w')| = \rho(d, d') \quad \square
\end{aligned}
$$

The second line is by Def. of $d_z$, while the third is by relaxing on absolute. The fourth is by properties of $w_z$. This lemma suggests that distance between progressed distributions would not increase.

**Proof of Lemma 1.** The proof of $(w_z)_{z'} = w_{z \cdot z'}$ is trivially by definition of $w_z$, here we prove $(e_z)_{z'} = e_{z \cdot z'}$. It suffices to prove that $e_{z \cdot a} = (e_z)_a$ for any $z \in \mathcal{Z}$, $a \in \mathcal{N}_{\mathcal{A}}$.

($\Rightarrow$) Suppose $d'' \in e_{z \cdot a}$, we show $d'' \in (e_z)_a$. Since $d'' \in e_{z \cdot a}$, $\exists \{d_1, d_2, \ldots, d_i, \ldots\}$ s.t. $d_i \in e$ and $\lim_{i \to \infty} (d_i)_{z \cdot a} = d''$. Consider another infinite sequence $\{d'_1, d'_2, \ldots, d'_i, \ldots\}$ where $d'_i = (d_i)_z$. Clearly $d'_i \in e_z$. Since $\lim_{i \to \infty} (d'_i)_a = \lim_{i \to \infty} (d_i)_{z \cdot a} = d''$, $d'' \in (e_z)_a$.

($\Leftarrow$) Conversely, suppose $d'' \in (e_z)_a$, then there exists an infinite sequence $\{d'_1, d'_2, \ldots d'_i, \ldots\}$ s.t. $d'_i \in e_z$ and $d'' = \lim_{i \to \infty} (d'_i)_a$. For any $i \in \mathbb{N}$, because $d'_i \in e_z$, there exists an infinite sequence of distributions $\{d_{i,1}, d_{i,2}, \ldots d_{i,j}, \ldots\}$ s.t. $d'_i = \lim_{j \to \infty} (d_{i,j})_z$. For any $i, j$:

$$
\begin{aligned}
\rho(d'', (d_{i,j})_{z \cdot a}) &\leq \rho(d'', (d'_i)_a) + \rho((d'_i)_a, (d_{i,j})_{z \cdot a}) && \text{(Triangle Inequality)} \\
&\leq \rho(d'', (d'_i)_a) + \rho(d'_i, (d_{i,j})_z) && \text{(Lemma 7)}
\end{aligned}
$$

Since $d'' = \lim_{i \to \infty} (d'_i)_a$ and $d'_i = \lim_{j \to \infty} (d_{i,j})_z$ for each $i$. We have

$$
\lim_{i,j \to \infty} \rho(d'', (d_{i,j})_{z \cdot a}) \leq \lim_{i,j \to \infty} (\rho(d'', (d'_i)_a) + \rho(d'_i, (d_{i,j})_z)) = 0
$$

Therefore, sequences converging to $d''$ can be found in $\{(d_{i,j})_{z \cdot a} | i, j \in \mathbb{N}\}$. Thus $d'' \in e_{z \cdot a}$. $\square$

### A.6. Proof of Theorem 6

**Proof of Theorem 6.** We prove by showing that for any formula $\alpha$ with free SO 0-ary variables among $\vec{F}$, $\models \exists\vec{F}.\alpha \equiv \exists\vec{v}.\alpha^{\vec{F}}_{\vec{v}}$, where $v_i$ are FO variables of the same sort as $F_i$ correspondingly. With Theorem 5, the theorem follows directly.

($\Rightarrow$) For any $e, w, z, \lambda$, $e, w, z, \lambda \models \exists\vec{F}.\phi$ iff (by semantics of $\exists\vec{F}$) $e, w, z, \lambda' \models \phi$ for some $\lambda'$ s.t. $\lambda' \sim_{F_i} \lambda$ for all $i$, hence $e, w, z \models \phi^{\vec{F}}_{\vec{n}}$ for $\vec{n}$ s.t. $n_i = \lambda'[F_i]$,[11] therefore (by semantics of $\exists\vec{v}$) $e, w, z \models \exists\vec{v}.\phi^{\vec{F}}_{\vec{v}}$, that is $e, w, z, \lambda \models \exists\vec{v}.\phi^{\vec{F}}_{\vec{v}}$.

($\Leftarrow$) For any $e, w, z, \lambda$, $e, w, z, \lambda \models \exists\vec{v}.\phi^{\vec{F}}_{\vec{v}}$ iff (by semantics of $\exists\vec{v}$) $e, w, z, \lambda \models \phi^{\vec{F}}_{\vec{n}}$ for some $\vec{n}$, then $e, w, z, \lambda' \models \phi$ for some $\lambda'$ s.t. $\lambda'[F_i] = n_i$ and $\lambda' \sim_{F_i} \lambda$ for all $i$, hence (by semantics of $\exists\vec{F}$) $e, w, z, \lambda \models \exists\vec{F}.\phi$. $\square$

### A.7. Proof of Lemma 2

**Proof of Lemma 2.** Suppose $e \models \boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma$, it suffices to show that $e_{t_{sen}} \models \boldsymbol{B}^{f'}$. By the semantics, we only need to show $e_{t_{sen}} \models \boldsymbol{B}(\vec{h} = \vec{n} : f'(\vec{n}))$ for all $\vec{n}$.

Let $e^\star_{t_{sen}} = \{d_z : d \in e, d \sim_{comp} z\}$. By definition $e_{t_{sen}} = cl(e^\star_{t_{sen}})$. For all $d_{t_{sen}} \in e^\star_{t_{sen}}$, we have

$$\begin{aligned}
\sum_{\{w: w\models \vec{h}=\vec{n}, w\in S(d_{t_{sen}})\}} d_{t_{sen}}(w) &= \sum_{\{w: w\models \vec{h}=\vec{n}, w\in S(d_{t_{sen}})\}} \sum_{\{w': w'\in S(d)\}} \sum_{\{t': t'\sim_{w'} t_{sen}, w'_{t'}=w\}} d(w') \times l^*(w', t') \\
&= \sum_{\{w: w\models \vec{h}=\vec{n}, w\in S(d_{t_{sen}})\}} \sum_{\{w': w'\in S(d), w'_{t_{sen}}=w\}} d(w') \times l^*(w', t_{sen}) = \sum_{\{w': w'\models [t_{sen}]\vec{h}=\vec{n}, w'\in S(d)\}} d(w') \times l^*(w', t_{sen}) \\
&= \sum_{\{w': w'\models \vec{h}=\vec{n}, w'\in S(d)\}} d(w') \times \mathcal{L}(t_{sen})^{\vec{h}}_{\vec{n}} \\
&= f(\vec{n}) \times c \times \mathcal{L}(t_{sen})^{\vec{h}}_{\vec{n}}
\end{aligned}$$

Here $c \in \mathbb{R}$ is a constant. The first equality is by definition of $d_{t_{sen}}$ while the second is due to that $t_{sen}$ is a sensing action, therefore it has no other alternatives but itself. The third equality is by switching the order of the first two summations and the definition of $w'_{t_{sen}}$. The fourth step is because $t_{sen}$ is a sensing and the fact $\{d\} \models \boldsymbol{K}\Sigma$. The last line is by the assumptions $d \in e$ and $e \models \boldsymbol{B}^f$, where $c$ is the number s.t. $\text{Eq}(d, \mathcal{W}_{\text{TRUE}}, c)$. The above procedure shows that $\text{Eq}(d_{t_{sen}}, \mathcal{W}_{h=n}, f(\vec{n}) \times c \times \mathcal{L}(t_{sen})^{\vec{h}}_{\vec{n}})$. In the same vein, one can prove that $\text{Eq}(d_{t_{sen}}, \mathcal{W}_{\text{TRUE}}, \eta \times c)$, where $\eta$ is as Theorem 7. Hence $\text{Norm}(d_{t_{sen}}, \mathcal{W}_{\vec{h}=\vec{n}}, f'(\vec{n}))$. This holds for all $d_{t_{sen}} \in e^\star_{t_{sen}}$ and all $\vec{n}$, which means $e^\star_{t_{sen}} \models \boldsymbol{B}^{f'}$. By Proposition 3 and Theorem 4, $e_{t_{sen}} \models \boldsymbol{B}^{f'}$. $\square$

### A.8. Proof of Lemma 3

**Proof of Lemma 3.** Let $\mathcal{W}_{cov}(d') \subseteq \mathcal{W}_{\mathcal{H}}$ be the minimal set which "covers" all worlds in $S(d')$, i.e. $\forall w \in S(d'), \exists \mathcal{C} \in \mathcal{W}_{cov}(d'), w \in \mathcal{C}$. Since $\{d'\} \models \boldsymbol{B}^{f'} \wedge \boldsymbol{K}\Sigma$, $S(d')$ must be countable and therefore $\mathcal{W}_{cov}(d')$ is countable. Thus there is an enumeration of elements in $\mathcal{W}_{cov}(d')$ as $\{\mathcal{C}'_1, \mathcal{C}'_2, ..., \}$. For each class $\mathcal{C}'_i \in \mathcal{W}_{cov}(d')$, we look for one class $\mathcal{C}_i \in \mathcal{W}_{\mathcal{H}}$ such that for any primitive terms $t \in \mathcal{P}_{\mathcal{H}}$, $w \in \mathcal{C}_i$ and action sequence $z$:

$$w[t, \langle\rangle] = w'[t, \langle\rangle] \text{ for all } w' \in \mathcal{C}' \quad \text{and} \quad w[t, t_{sen} \cdot z] = w'[t, z] \text{ for all } w' \in \mathcal{C}'$$

Such a class $\mathcal{C}$ exists.[12] It is not hard to prove that $\mathcal{C}_i$ defined in this way have the following properties:

$$\forall i. w \in \mathcal{C}_i \text{ entails } w_{t_{sen}} \in \mathcal{C}'_i \quad \text{and} \quad \forall i.\forall j. i \neq j \text{ entails } \mathcal{C}_i \neq \mathcal{C}_j \tag{A.1}$$

Now we construct the distribution $d$ as follows: for any world $w$,

$$d(w) = \begin{cases} d'(w_{t_{sen}})/\mathcal{L}(t_{sen})^{\vec{h}}_{\vec{n}} & w = w_{\vec{n},\mathcal{C}} \text{ for some } \vec{n}, \mathcal{C} \in \{\mathcal{C}_1, \mathcal{C}_2, ...\}, d'(w_{t_{sen}}) > 0 \\ 0 & o.w. \end{cases} \tag{A.2}$$

It can be proved that (A.2) is well-defined: when $d'(w_{t_{sen}}) > 0$, we have $f'(\vec{n}) > 0$, according to the definition of $f'$, $\mathcal{L}(t_{sen})^{\vec{h}}_{\vec{n}} > 0$. Now we show that $\{d\} \models \boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma$ and $d_{t_{sen}} = d'$. Obviously $\{d\} \models \boldsymbol{K}\Sigma$ (since $d(w) = 0$ for any $w \not\models \Sigma$). To prove $\{d\} \models \boldsymbol{B}^f$, it suffices to prove that for all $\vec{n}$, $\sum_{\{w: w\in S(d), w\models \vec{h}=\vec{n}\}} d(w) / \sum_{w\in S(d)} d(w) = f(\vec{n})$.

[11] $\lambda[F]$ stands for the denotation of SO primitive term $F$ under $\lambda$.

[12] For each $\mathcal{C}' \in \mathcal{W}_{cov}(d')$, there could be infinitely many classes which satisfy the two conditions, we randomly select one of them.

Let $\eta$ be as in Theorem 7. For the numerator $\sum_{\{w:w\in S(d),w\models\vec{h}=\vec{n}\}} d(w)$, we have

$$
\begin{aligned}
&\sum_{\{w:w\in S(d),w\models\vec{h}=\vec{n}\}} d(w) = \sum_{\mathcal{C}\in\{\mathcal{C}_1,\mathcal{C}_2,\dots\}} \sum_{\{w:w\models\vec{h}=\vec{n} \text{ and } w\in\mathcal{C}\}} d(w) \\
&= \sum_{\mathcal{C}\in\{\mathcal{C}_1,\mathcal{C}_2,\dots\}} \sum_{\{w:w\models\vec{h}=\vec{n} \text{ and } w\in\mathcal{C}\}} \left(d'(w_{t_{sen}})/\mathcal{L}(t_{sen})^{\vec{h}}_{\vec{n}}\right) \quad \text{(By Eq. (A.2))} \\
&= \sum_{\mathcal{C}'\in\{\mathcal{C}'_1,\mathcal{C}'_2,\dots\}} \sum_{\{w':w'\models\vec{h}=\vec{n} \text{ and } w'\in\mathcal{C}'\}} \left(d'(w')/\mathcal{L}(t_{sen})^{\vec{h}}_{\vec{n}}\right) \quad \text{(By Proposition 4)} \\
&= \sum_{\{w':w'\in S(d'),w'\models\vec{h}=\vec{n}\}} d'(w')/\mathcal{L}(t_{sen})^{\vec{h}}_{\vec{n}} \quad \text{(By Proposition 4)} \\
&= f'(\vec{n})\cdot c/\mathcal{L}(t_{sen})^{\vec{h}}_{\vec{n}} \quad (\text{since } \{d'\}\models \boldsymbol{B}^{f'},\ \text{EQ}(d',\mathcal{W}_{\text{TRUE}},c) \text{ for some } c) \\
&= \frac{f(\vec{n})}{\eta}\cdot c \quad (\text{By Def. of } f')
\end{aligned}
$$

Likewise, one can prove $\sum_{w\in S(d)} d(w) = \frac{c}{\eta}$, therefore $\sum_{\{w:w\in S(d),w\models\vec{h}=\vec{n}\}} d(w)/\sum_{w\in S(d)} d(w) = f(\vec{n})$.

The task remains to prove $d_{t_{sen}} = d'$. It is equivalent to prove that for any $w'$,

$$
\sum_{\{w:w\in S(d),\,w_{t_{sen}}=w'\}} d(w)\times l^*(w,t_{sen}) = d'(w') \tag{A.3}
$$

By the construction of $d$, $\forall w'$ with $d'(w')=0$, if $w_{t_{sen}}=w'$, then $d(w)=0$. For $w'$ with $d'(w')>0$, w.l.o.g. supposing that $w'=w_{\vec{n},\mathcal{C}'_i}$ for some $\vec{n},\mathcal{C}'_i$,

$$
\begin{aligned}
\text{LHS} &= \sum_{\mathcal{C}\in\{\mathcal{C}_1,\mathcal{C}_2,\dots\}} \sum_{\vec{u}} \sum_{\{w:w_{n_a}=w',w\in\mathcal{C},w\models\vec{h}=\vec{u}\}} d(w)\times l^*(w,t_{sen}) \\
&= \sum_{\mathcal{C}\in\{\mathcal{C}_1,\mathcal{C}_2,\dots\}} \sum_{\{w:w_{t_{sen}}=w',w\in\mathcal{C},w\models\vec{h}=\vec{n}\}} d(w)\times l^*(w,t_{sen}) \\
&= \sum_{\{w:w_{t_{sen}}=w',w\in\mathcal{C}_i,w\models\vec{h}=\vec{n}\}} d(w)\times l^*(w,t_{sen}) \quad (\text{Property in Eq. (A.1)},\ w'\in\mathcal{C}'_i) \\
&= \sum_{\{w:w_{t_{sen}}=w',w\in\mathcal{C}_i,w\models\vec{h}=\vec{n}\}} d(w)\times \mathcal{L}(t_{sen})^{\vec{h}}_{\vec{n}} \quad (\text{Def. } \Sigma_l) \\
&= \frac{d'(w')}{\mathcal{L}(t_{sen})^{\vec{h}}_{\vec{n}}}\times \mathcal{L}(t_{sen})^{\vec{h}}_{\vec{n}} \quad \text{(By Proposition 4 and Eq. (A.2))} \\
&= d'(w')
\end{aligned}
$$

□

### *A.9. Proof of Lemma 5*

We begin with the following two propositions:

**Proposition 7.** *Given a BAT $\Sigma$, action $t_a$, world $w$ s.t. $w\models\vec{h}=\vec{n}\wedge\Sigma$ for some $\vec{n}$, let $\mathbb{I}$ be as in Theorem 8, then for all $\vec{n}'$ and $t_{a'}$, $t_{a'}\sim_w t_a$ and $w\models[t_a]\vec{h}=\vec{n}'$ iff $w\models\mathbb{I}(\vec{n}',\vec{n},t_{a'},t_a)=1$.*

**Proof.** $t_{a'}\sim_w t_{a'}$ and $w\models[t_{a'}]\vec{h}=\vec{n}'$ iff (since $w\models\Sigma$)

$w\models(\Psi)^{a,a'}_{t_a,t_{a'}}$ and $w\models[t_a]\vec{h}=\vec{n}'$ iff (By Def. of $Pro$ and $w\models\vec{h}=\vec{n}\wedge\Sigma$)

$w\models(\Psi)^{a,a'}_{t_a,t_{a'}}$ and $w_{t_{a'}}\models\vec{h}=\vec{n}'\wedge\Sigma\wedge Pro(\vec{h}=\vec{n},t_{a'})$ iff

$w\models(\Psi)^{a,a'}_{t_a,t_{a'}}$ and $w_{t_{a'}}\models Pro(\vec{h}=\vec{n},t_{a'})^{\vec{h}}_{\vec{n}'}$ iff (since $Pro(\vec{h}=\vec{n},t_{a'})^{\vec{h}}_{\vec{n}'}$ is rigid)

$w\models(\Psi)^{a,a'}_{t_a,t_{a'}}$ and $w\models Pro(\vec{h}=\vec{n},t_{a'})^{\vec{h}}_{\vec{n}'}$ iff (by Def. of $\mathbb{I}$) $w\models\mathbb{I}(\vec{n}',\vec{n},t_{a'},t_a)=1$. □

**Proposition 8.** *Given $d\in\mathcal{D}$ and stochastic action $a\in\mathcal{N}_A$, $\sum_{(w',a')\in\mathcal{W}^{\{d\},a}_{\text{TRUE}}} d(w')\times l^*(w',a') = \sum_{w\in\mathcal{W}^{\{d\},\langle\rangle}_{\text{TRUE}}} d(w)$.*

**Proof.** Note that $\mathcal{W}^{\{d\},a}_{\text{TRUE}} = \{(w',a'):a'\sim_{w'}a\}$ and $l^*(w',a')=w'[l(a),\langle\rangle]$, we have

$\sum_{(w',a')\in\mathcal{W}^{\{d\},a}_{\text{TRUE}}} d(w')\times l^*(w',a') = \sum_{\{(w',a'):a'\sim_{w'}a\}} d(w')\times w'[l(a),\langle\rangle]$

$= \sum_{w'\in S(d)}\sum_{\{a':a'\sim_{w'}a\}} d(w')\times w'[l(a),\langle\rangle] = \sum_{w'\in S(d)} d(w')\times\sum_{\{a':a'\sim_{w'}a\}} w'[l(a),\langle\rangle]$

$= \sum_{w'\in S(d)} d(w') = \sum_{w\in\mathcal{W}^{\{d\},\langle\rangle}_{\text{TRUE}}} d(w)$

The second line is because $w[l(a), z] \leq 1$ and $\sum_{\{a':a' \sim_w a\}} w[l(a), z] = 1$, for all $w \in \mathcal{W}, z \in \mathcal{Z}$, and $a \in \mathcal{N}_A$. $\square$

We are now ready to prove Lemma 5.

**Proof of Lemma 5.** Now, supposing $d' \in \mathcal{D}$ and $\{d'\} \models \boldsymbol{B}^{f'} \wedge \Sigma$, w.l.o.g. we assume that $d'$ is normalized, i.e. $\sum_{w'} d'(w') = 1$. In case $\sum_{w'} d'(w') = c$, for $c \neq 1$, a distribution can be constructed in the same way except that the weight of worlds is proportionally increased by $c$.

Let $\mathcal{W}_{\text{cov}}(d') = \{\mathcal{C}'_1, \mathcal{C}'_2, ..., \mathcal{C}'_n, ...\}$. It is obvious that the set of alternatives of $t_{sa}$ is countable. We denote the set of alternatives as $\{a_1, a_2, ..., a_k, ...\}$. Given a world $w$ s.t. $w \models \vec{h} = \vec{n} \wedge \Sigma$, due to the finite alternative hypotheses, there are only finite alternatives $\{a_{\vec{n},1}, ..., a_{\vec{n},m}\}$ whose likelihoods are positive. Moreover, $w$ might progress to $m$ worlds $w_{a_{\vec{n},1}}, w_{a_{\vec{n},2}}, \ldots, w_{a_{\vec{n},m}}$ with equivalence class $\mathcal{C}'_{a_{\vec{n},1}}, \mathcal{C}'_{a_{\vec{n},2}}, \ldots, \mathcal{C}'_{a_{\vec{n},m}}$. Conversely, given $m, \vec{n}$ and equivalence class $\mathcal{C}'_{a_{\vec{n},1}}, \mathcal{C}'_{a_{\vec{n},2}}, \ldots, \mathcal{C}'_{a_{\vec{n},m}}$, there exists a $w$ s.t. $w \models \vec{h} = \vec{n} \wedge \Sigma$ and $w_{a_{\vec{n},i}} \in \mathcal{C}'_{a_{\vec{n},i}}$ for any $i \in \{1, ..., m\}$

Now consider a distribution $d$ as follows:

$$d(w) = \begin{cases} f(\vec{n}) \prod_{i=1}^{m} \frac{d'(w_{a_{\vec{n},i}})}{f'(\vec{n}'_i)} & w = w_{\vec{n},\mathcal{C}'_{a_{\vec{n},1}},\mathcal{C}'_{a_{\vec{n},2}}\ldots\mathcal{C}'_{a_{\vec{n},m}}} \text{ for some } \vec{n}, \mathcal{C}'_{a_{\vec{n},1}}, \mathcal{C}'_{a_{\vec{n},2}}, \ldots, \mathcal{C}'_{a_{\vec{n},m}} \text{ and } w \models \bigwedge_i [a_{\vec{n},i}]\vec{h} = \vec{n}'_i \\ & \text{for some } \vec{n}'_1, \vec{n}'_2, \ldots, \vec{n}'_m \text{ and } d'(w_{a_{\vec{n},i}}) > 0 \text{ for all } i \\ 0 & o.w. \end{cases} \tag{A.4}$$

Note that $w_{\vec{n},\mathcal{C}_{a_{\vec{n},1}},\mathcal{C}_{a_{\vec{n},m}}} = w_{\vec{n}',\mathcal{C}'_{a_{\vec{n}',1}},\mathcal{C}'_{a_{\vec{n}',m}}}$ iff $\vec{n} = \vec{n}'$ and $\mathcal{C}_{a_{\vec{n},i}} = \mathcal{C}'_{a_{\vec{n}',i}}$ for all $i$. Therefore, for each $w$ who gains the assignment, there is a unique combination of $\vec{n}, \mathcal{C}'_{a_{\vec{n},1}}, \ldots \mathcal{C}'_{a_{\vec{n},m}}$ corresponding to it. Additionally, $w_{a_{\vec{n},i}} = w_{\mathcal{C}'_{a_{\vec{n},i}},\vec{n}'_i}$. We claim that the above constructed distribution $d$ satisfies $d_{t_{sa}} = d'$ and $\{d\} \models \boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma$.

For $d_{t_{sa}} = d'$, for any world $w'$, there are two cases:

**Case 1** $w' \notin S(d')$: suppose $w' \in \mathcal{C}'$ for some $\mathcal{C}'$. With Eq (A.4) it is not hard to prove that $d(w) = 0$ for any $w$ and $a$ s.t. $w_a = w'$. Therefore $d_{t_{sa}}(w') = d'(w') = 0$.

**Case 2** $w' \in S(d')$: w.l.o.g. supposing $w' \in \mathcal{C}'$ and $w' \models \vec{h} = \vec{n}'$ for some $\mathcal{C}' \in \mathcal{W}_{\text{cov}}(d')$ and $\vec{n}'$, in the following, we use $a \sim t_{sa}$ to mean action $a$ is an alternative of $t_{sa}$,

$$\begin{aligned}
d_{t_{sa}}(w') &= \sum_{\{a:a\sim t_{sa}\}} \sum_{\{w:w\in S(d) \text{ and } w_a = w'\}} d(w) \times l^*(w,a) \quad (\text{Def. of } d_z) \\
&= \sum_{\{a:a\sim t_{sa}\}} \sum_{\vec{u}} \sum_{\{w:w\in S(d) \text{ and } w_a = w' \text{ and } w\models\vec{h}=\vec{u}\}} d(w) \times l^*(w,a) \quad (\text{Equivalent Transformation}) \\
&= \sum_{\{a:a\sim t_{sa}\}} \sum_{\vec{u}} \sum_{\{w:w\in S(d) \text{ and } w_a \in \mathcal{C}' \text{ and } w\models\vec{h}=\vec{u}\wedge[a]\vec{h}=\vec{n}'\}} d(w) \times l^*(w,a) \quad (\text{By Proposition 4}) \\
&= \sum_{a} \sum_{\vec{u}} \mathbb{I}(\vec{n}', \vec{u}, a, t_{sa}) \sum_{\{w:w_a\in\mathcal{C}' \text{ and } w\models\vec{h}=\vec{u}\}} d(w) \times l^*(w,a) \quad (\text{By Proposition 7}) \\
&= \sum_{a} \sum_{\vec{u}} \mathbb{I}(\vec{n}', \vec{u}, a, t_{sa}) \times \mathcal{L}(a)_{\vec{u}}^{\vec{h}} \times \sum_{\{w:w_a\in\mathcal{C}' \text{ and } w\models\vec{h}=\vec{u}\}} d(w) \quad (\text{By Def. } \Sigma_l\ ) \\
&= \sum_{i=1}^{m} \sum_{\vec{u}} \mathbb{I}(\vec{n}', \vec{u}, a_{\vec{u},i}, t_{sa}) \times \mathcal{L}(a)_{\vec{u}}^{\vec{h}} \times \sum_{\mathcal{C}'_{a_{\vec{u},1}},\ldots\mathcal{C}'_{a_{\vec{u},i-1}},\mathcal{C}'_{a_{\vec{u},i+1}},\ldots,\mathcal{C}'_{a_{\vec{u},m}}} d(w_{\vec{u},\mathcal{C}'_{a_{\vec{u},1}},\ldots,\mathcal{C}'_{a_{\vec{u},i-1}},\mathcal{C}',\mathcal{C}'_{a_{\vec{u},i+1}},\ldots,\mathcal{C}'_{a_{\vec{u},m}}}) \quad (\#1) \\
&= \sum_{i=1}^{m} \sum_{\vec{u}} \mathbb{I}(\vec{n}', \vec{u}, a_{\vec{u},i}, t_{sa}) \times \mathcal{L}(a)_{\vec{u}}^{\vec{h}} \times \sum_{\mathcal{C}'_{a_{\vec{u},1}},\ldots\mathcal{C}'_{a_{\vec{u},i-1}},\mathcal{C}'_{a_{\vec{u},i+1}},\ldots,\mathcal{C}'_{a_{\vec{u},m}}} f(\vec{u}) \prod_{j=1}^{m} \sum_{\{\vec{u}':\Sigma\models\vec{h}=\vec{u}\supset[a_{\vec{u},j}]\vec{h}=\vec{u}'\}} \frac{d'(w_{a_{\vec{u},j}})}{f'(\vec{u}')} \quad (\#2) \\
&= \sum_{i=1}^{m} \sum_{\vec{u}} \mathbb{I}(\vec{n}', \vec{u}, a_{\vec{u},i}, t_{sa}) \times \mathcal{L}(a)_{\vec{u}}^{\vec{h}} \times \sum_{\mathcal{C}'_{a_{\vec{u},1}},\ldots\mathcal{C}'_{a_{\vec{u},i-1}},\mathcal{C}'_{a_{\vec{u},i+1}},\ldots,\mathcal{C}'_{a_{\vec{u},m}}} f(\vec{u}) \prod_{j=1}^{m} \sum_{\{\vec{u}':\Sigma\models\vec{h}=\vec{u}\supset[a_{\vec{u},j}]\vec{h}=\vec{u}'\}} \frac{d'(w_{\mathcal{C}'_{a_{\vec{u},j}},\vec{u}'})}{f'(\vec{u}')} \quad (\#3) \\
&= \sum_{i=1}^{m} \sum_{\vec{u}} \mathbb{I}(\vec{n}', \vec{u}, a_{\vec{u},i}, t_{sa}) \times \mathcal{L}(a)_{\vec{u}}^{\vec{h}} \times \sum_{\mathcal{C}'_{a_{\vec{u},1}},\ldots\mathcal{C}'_{a_{\vec{u},i-1}},\mathcal{C}'_{a_{\vec{u},i+1}},\ldots,\mathcal{C}'_{a_{\vec{u},m}}} f(\vec{u}) \prod_{j=1,j\neq i}^{m} \sum_{\{\vec{u}':\Sigma\models\vec{h}=\vec{u}\supset[a_{\vec{u},j}]\vec{h}=\vec{u}'\}} \frac{d'(w_{\mathcal{C}'_{a_{\vec{u},j}},\vec{u}'})}{f'(\vec{u}')} \times \frac{d'(w_{\mathcal{C}',\vec{n}'})}{f'(\vec{n}')} \quad (\#4) \\
&= \sum_{i=1}^{m} \sum_{\vec{u}} \mathbb{I}(\vec{n}', \vec{u}, a_{\vec{u},i}, t_{sa}) \times \mathcal{L}(a)_{\vec{u}}^{\vec{h}} \times f(\vec{u}) \times \frac{d'(w_{\mathcal{C}',\vec{n}'})}{f'(\vec{n}')} \sum_{\mathcal{C}'_{a_{\vec{u},1}},\ldots\mathcal{C}'_{a_{\vec{u},i-1}},\mathcal{C}'_{a_{\vec{u},i+1}},\ldots,\mathcal{C}'_{a_{\vec{u},m}}} \prod_{j=1,j\neq i}^{m} \sum_{\{\vec{u}':\Sigma\models\vec{h}=\vec{u}\supset[a_{\vec{u},j}]\vec{h}=\vec{u}'\}} \frac{d'(w_{\mathcal{C}'_{a_{\vec{u},j}},\vec{u}'})}{f'(\vec{u}')} \quad (\#5)
\end{aligned}$$

$$=\sum_{i=1}^{m}\sum_{\vec{u}}\mathbb{I}(\vec{n}',\vec{u},a_{\vec{u},i},t_{sa})\times\mathcal{L}(a)_{\vec{u}}^{\vec{h}}\times f(\vec{u})\times\frac{d'(w_{\mathcal{C}',\vec{n}'})}{f'(\vec{n}')}\times 1 \quad (\#6)$$

$$=\frac{d'(w_{\vec{n}',\mathcal{C}'})}{f'(\vec{n}')}f'(\vec{n}')=d'(w_{\vec{n}',\mathcal{C}'})=d'(w')$$

here, $\sum_{\mathcal{C}'_{a_{\vec{u},1}},\dots\mathcal{C}'_{a_{\vec{u},i-1}},\mathcal{C}'_{a_{\vec{u},i+1}},\dots,\mathcal{C}'_{a_{\vec{u},m}}}$ is an abbr. for $\sum_{\mathcal{C}'_{a_{\vec{u},1}}\in\mathcal{W}_{\text{cov}}(d')}\cdots\sum_{\mathcal{C}'_{a_{\vec{u},i-1}}\in\mathcal{W}_{\text{cov}}(d')}\sum_{\mathcal{C}'_{a_{\vec{u},i+1}}\in\mathcal{W}_{\text{cov}}(d')}\cdots\sum_{\mathcal{C}'_{a_{\vec{u},m}}\in\mathcal{W}_{\text{cov}}(d')}$.

The idea in step #1 is basically to partition worlds $w$ which satisfy $w_a\in\mathcal{C}'$ and $w\models\vec{h}=\vec{u}$ according to the equivalence class $\mathcal{C}'_{a_{\vec{n},i}}$ that it might progress to through action $a_{\vec{n},i}$. Step #2 is by our construction of $d$, noting that for each action $a_{\vec{n},j}$, the progressed world $w_{a_{\vec{n},j}}$ might satisfy $\vec{h}=\vec{n}'_j$ for a unique $\vec{n}'_j$ and this $\vec{n}'_j$ is captured by the sum $\sum_{\{\vec{u}':\Sigma\models\vec{h}=\vec{u}\supset[a_{\vec{u},j}]\vec{h}=\vec{u}'\}}$. Step #3 and step #4 are because $w_{a_{\vec{n},j}}=w_{\mathcal{C}'_{a_{\vec{n},j}},\vec{u}'_i}$, additionally if $j=i$ we have $w_{a_{\vec{n},i}}=w_{\mathcal{C}',\vec{n}'}$ (hence $f'(\vec{n}')$ is used instead of $f'(\vec{u}')$ in the denominator) by assumption. In step #5, common terms are extracted to the left. Step #6 is due to the fact that $\sum_{\mathcal{C}'_{a_{\vec{u},j}}}d'(w_{\mathcal{C}'_{a_{\vec{u},j}},\vec{u}'})=f'(\vec{u}')$ for all $j,\vec{u}'$, applying this for multiple times, the "$\sum\prod\sum$" on the right of the equation in step #5 will eventually reduce to 1.

Now we prove that $\{d\}\models\boldsymbol{B}^f\wedge\boldsymbol{K}\Sigma$. By Proposition 8 and the assumption that $\sum_{w'}d'(w')=1$, it suffices to prove that for any $\vec{n}$, $\sum_{\{w:w\in S(d),w\models\vec{h}=\vec{n}\wedge\Sigma\}}d(w)=f(\vec{n})$.

$$\text{LHS}=\sum_{\{w:w\in S(d),w\models\vec{h}=\vec{n}\wedge\Sigma\}}d(w)=\sum_{\mathcal{C}'_{a_{\vec{n},1}},\dots\mathcal{C}'_{a_{\vec{n},m}}}d(w_{\vec{n},\mathcal{C}'_{a_{\vec{n},1}},\dots\mathcal{C}'_{a_{\vec{n},m}}}) \quad \text{(By Def. of } d)$$

$$=\sum_{\mathcal{C}'_{a_{\vec{n},1}},\dots\mathcal{C}'_{a_{\vec{n},m}}}f(\vec{n})\prod_{i=1}^{m}\sum_{\{\vec{u}':\Sigma\models\vec{h}=\vec{n}\supset[a_{\vec{n},i}]\vec{h}=\vec{u}'\}}\frac{d'(w_{\vec{u}',\mathcal{C}'_{a_{\vec{n},i}}})}{f'(\vec{u}')} \quad \text{(Equivalent Transformation)}$$

$$=f(\vec{n})\sum_{\mathcal{C}'_{a_{\vec{n},1}},\dots\mathcal{C}'_{a_{\vec{n},m}}}\prod_{i=1}^{m}\sum_{\{\vec{u}':\Sigma\models\vec{h}=\vec{n}\supset[a_{\vec{n},i}]\vec{h}=\vec{u}'\}}\frac{d'(w_{\vec{u}',\mathcal{C}'_{a_{\vec{n},i}}})}{f'(\vec{u}')} \quad (\#1)$$

$$=f(\vec{n})\times 1=\text{RHS}$$

Again, step #1 is due to the fact that $\sum_{\mathcal{C}'_{a_{\vec{u},j}}}d'(w_{\mathcal{C}'_{a_{\vec{u},j}},\vec{u}'})=f'(\vec{u}')$. □

### A.10. Proof of Lemma 6

To prove the lemma, we need the following concepts.

**Definition 13.** A sequence of functions $\{f_1,f_2,\dots,f_i,\dots\}$ uniformly converges to $f$ iff for any $\epsilon>0$ there is an $N\in\mathbb{N}$ s.t. for any $i>N$ and $\vec{u}$, $|f_i(\vec{u})-f(\vec{u})|<\epsilon$.

In the rest of the paper, suppose $\{a_1,a_2,...,a_k,...\}$ is an enumeration of alternatives of stochastic action $t_{sa}$.

**Definition 14.** Given $\boldsymbol{B}^f$ and $t_{sa}$ as in Theorem 8. For each $N\in\mathbb{N}$, we define function $f'_i$ as

$$f'_i(\vec{u}')=\sum_{j=1}^{i}\sum_{\vec{u}}f(\vec{u})\times\mathcal{L}(a_j)_{\vec{u}}^{\vec{h}}\times\mathbb{I}(\vec{u},\vec{u}',a_j,t_{sa})$$

Intuitively, $f'_i$ is the summation of contribution wrt the first $i$ actions from $f$ to $f'$.

**Lemma 8** *(Uniform convergence). Given $\boldsymbol{B}^f$, $\boldsymbol{B}^{f'}$, $\Sigma$ and stochastic $t_{sa}$ as in Theorem 8, given $f'_i$ as in Definition 14, then the sequence $\{f'_1,f'_2,\dots,f'_i,\dots\}$ uniformly converges to $f'$, i.e. for any $\epsilon$, there exists a $N>0$ s.t. for any $i>N$ and $\vec{u}'$, $|f'_i(\vec{u}')-f'(\vec{u}')|<\epsilon$.*

**Proof.** Since $\sum_{\vec{u}'}f'(\vec{u}')=1$ and $f'(\vec{u}')\ge 0$, given $\epsilon$, for all $\vec{u}'$, there are only finitely many values $\{\vec{u}'_1,\vec{u}'_2,\dots,\vec{u}'_n\}$ such that $f'(\vec{u}'_i)\ge\epsilon$.

In case $\vec{u}'\notin\{\vec{u}'_1,\vec{u}'_2,\dots,\vec{u}'_n\}$, $|f'_i(\vec{u}')-f'(\vec{u}')|<f'(\vec{u}')\le\epsilon$.

In case $\vec{u}'\in\{\vec{u}'_1,\vec{u}'_2,\dots,\vec{u}'_n\}$, since $\lim_{i\to\infty}f'_i(\vec{u}')=f'(\vec{u}')$, for each $\vec{u}'_j\in\{\vec{u}'_1,\vec{u}'_2,\dots,\vec{u}'_n\}$ there must be an integer $N_j$ s.t. for all $i>N_j$, $|f'_i(\vec{u}'_j)-f'(\vec{u}'_j)|<\epsilon$. Then, let $N=\max\{N_1,...,N_n\}$, clearly $|f'_i(\vec{u}')-f'(\vec{u}')|<\epsilon$ for all $i>N$ □

The lemma indicates that $f'$ can be approximated arbitrary closely by partial summation wrt finitely many alternatives.

**Proposition 9.** *Given $\boldsymbol{B}^f$, $\boldsymbol{B}^{f'}$, $\Sigma$ and $t_{sa}$ as in Theorem 8, then for any $\epsilon$, there exists a $N > 0$ s.t. for any $i > N$ and $\vec{u}'$, $\left(\sum_{j>i}\sum_{\vec{u}} f(\vec{u}) \times \mathcal{L}(a_j)^{\vec{h}}_{\vec{u}} \times \mathbb{I}(\vec{u},\vec{u}',a_j,t_{sa})\right) < \epsilon$.*

**Proof.** By Lemma 8 we know that for given $\epsilon$, there exists a $N > 0$ s.t. for any $i > N$ and $\vec{u}'$, $|f'_i(\vec{u}') - f'(\vec{u}')| < \epsilon$ where $f'_i$ is as in Definition 14. By the definition of $f'$ and $f'_i$ we have

$$\begin{aligned}
&|f'_i(\vec{u}') - f'(\vec{u}')| \\
=&|\sum_{j=1}^{i}\sum_{\vec{u}} f(\vec{u}) \times \mathcal{L}(a_j)^{\vec{h}}_{\vec{u}} \times \mathbb{I}(\vec{u},\vec{u}',a_j,t_{sa}) - \sum_{j}\sum_{\vec{u}} f(\vec{u}) \times \mathcal{L}(a_j)^{\vec{h}}_{\vec{u}} \times \mathbb{I}(\vec{u},\vec{u}',a_j,t_{sa})| \\
=&|\sum_{j>i}\sum_{\vec{u}} f(\vec{u}) \times \mathcal{L}(a_j)^{\vec{h}}_{\vec{u}} \times \mathbb{I}(\vec{u},\vec{u}',a_j,t_{sa})| = \sum_{j>i}\sum_{\vec{u}} f(\vec{u}) \times \mathcal{L}(a_j)^{\vec{h}}_{\vec{u}} \times \mathbb{I}(\vec{u},\vec{u}',a_j,t_{sa}) < \epsilon
\end{aligned}$$

□

Now, we are ready to prove Lemma 6.

**Proof of Lemma 6.** Given $\{d'\} \models \boldsymbol{B}^{f'} \wedge \boldsymbol{K}\Sigma$, suppose that $\text{EQ}(d', \mathcal{W}_{\text{TRUE}}, c)$ for some $c \in \mathbb{R}$. For arbitrary $\epsilon$, let $\{w'_1, w'_2, ...\}$ be an enumeration of $S(d')$. Since $\text{EQ}(d', \mathcal{W}_{\text{TRUE}}, c)$, there exists a number $N$ s.t. $\sum_{w' \in \{w'_i : i > N\}} d'(w') < \frac{\epsilon}{4}$. For each $w'_i$ with $i \leq N$, let $\vec{n}'_i$ be the standard names s.t. $w'_i \models \vec{h} = \vec{n}'_i$. We define $\nabla = \min\{f'(\vec{n}'_1), ..., f'(\vec{n}'_N)\}$. Obviously,

$$\text{For } i \leq N, \nabla \leq f'(\vec{n}'_i) \tag{A.5}$$

and $0 < \nabla \leq 1$ (the "<" is due to $w'_i \in S(d')$).

By Proposition 9, there is a $M$ such that for any $M' \geq M$ and $\vec{u}'$

$$\sum_{j>M'}\sum_{\vec{u}} f(\vec{u}) \times \mathcal{L}(a_j)^{\vec{h}}_{\vec{u}} \times \mathbb{I}(\vec{u},\vec{u}',a_j,t_{sa}) < \frac{\nabla\epsilon}{2cN} \tag{A.6}$$

For $w' = w'_i$ with $i \leq N$, let $\vec{n}'_i$ be defined as above. Since $w'_i \in S(d')$, $d'(w'_i) > 0$ and $f'(\vec{n}_i) > 0$. By the definition of $f'$, there exist $a, \vec{u}$ such that $\mathcal{L}(a)^{\vec{h}}_{\vec{u}} \times \mathbb{I}(\vec{n}', \vec{u}_i, a, t_{sa}) > 0$. Let $M_i$ be the smallest index in the enumeration $\{a_1, \ldots a_j, \ldots\}$ s.t. $\mathcal{L}(a_{M_i})^{\vec{h}}_{\vec{u}} \times \mathbb{I}(\vec{n}', \vec{u}_i, a_{M_i}, t_{sa}) > 0$ for some $\vec{u}$. Basically it means that $w_i$ is the progressed world of some worlds w.r.t. action $a_{M_i}$.

Now, we consider the first $m$ alternatives of $t_{sa}$ in $\{a_1, \ldots a_j, \ldots\}$ where $m = \max\{M, M_1, ..., M_N\}$. Similar to the proof of Lemma 5, we construct a distribution $d$ wrt $d'$ and the finite alternatives $\{a_1, a_2, \ldots a_m\}$ as follows:

$$d(w) = \begin{cases} cf(\vec{n})\prod_{i=1}^{m}\frac{d'(w_{\mathcal{C}'_i,\vec{n}'_i})}{cf'(\vec{n}'_i)} & w = w_{\vec{n},\mathcal{C}'_1,\mathcal{C}'_2,...,\mathcal{C}'_m} \text{ for some } \vec{n}, \mathcal{C}'_1, \mathcal{C}'_2, ..., \mathcal{C}'_m \text{ and } w \models \bigwedge_i [a_i]\vec{h} = \vec{n}'_i \\ & \text{for some } \vec{n}'_1, \vec{n}'_2, ..., \vec{n}'_m, d'(w_{a_i}) > 0 \text{ for } i \in \{1, \ldots m\} \\ 0 & o.w. \end{cases} \tag{A.7}$$

Similar to the proof of Lemma 5, one can prove $\{d\} \models \boldsymbol{B}^f \wedge \boldsymbol{K}\Sigma$. Now, we show $\rho(d_{t_{sa}}, d') < \epsilon$:

For any $w' = w'_i$, with $i \leq N$, w.l.o.g. supposing $w' = w_{\mathcal{C}',\vec{n}'}$ for some $\mathcal{C}', \vec{n}'$, i.e. $w \in \mathcal{C}'$ and $w' \models \vec{h} = \vec{n}' \wedge \Sigma$:

Similar to the proof of Lemma 5, we have

$$\begin{aligned}
d_{t_{sa}}(w') = &\sum_{i\leq m}\sum_{\vec{u}} \mathbb{I}(\vec{n}', \vec{u}, a_i, t_{sa}) \times \mathcal{L}(a_i)^{\vec{h}}_{\vec{u}} \times \sum_{\{w : w_{a_i} \in \mathcal{C}' \text{ and } w \models \vec{h} = \vec{u}\}} d(w) + \\
&\sum_{i>m}\sum_{\vec{u}} \mathbb{I}(\vec{n}', \vec{u}, a_i, t_{sa}) \times \mathcal{L}(a_i)^{\vec{h}}_{\vec{u}} \times \sum_{\{w : w_{a_i} \in \mathcal{C}' \text{ and } w \models \vec{h} = \vec{u}\}} d(w)
\end{aligned}$$

We consider the partial summation over $i \leq m$ and $i > m$ separately and denote them respectively as $P_1$, $P_2$, namely $d_{t_{sa}}(w') = P_1 + P_2$.

For $P_1$, similar to the proof of the finite alternatives case, we have

$$\begin{aligned}
P_1 = &\sum_{i\leq m}\sum_{\vec{u}} \mathbb{I}(\vec{n}', \vec{u}, a_i, t_{sa}) \times \mathcal{L}(a_i)^{\vec{h}}_{\vec{u}} \times \sum_{\{w : w_{(a_i)} = w' \text{ and } w \models \vec{h} = \vec{u}\}} d(w) \\
= &\sum_{i\leq m}\sum_{\vec{u}} \mathbb{I}(\vec{n}', \vec{u}, a_i, t_{sa}) \times \mathcal{L}(a_i)^{\vec{h}}_{\vec{u}} \times \sum_{\mathcal{C}'_1,...,\mathcal{C}'_{i-1},\mathcal{C}'_{i+1},...,\mathcal{C}'_m} d(w_{\vec{u},\mathcal{C}'_1,...,\mathcal{C}'_{i-1},\mathcal{C}',\mathcal{C}'_{i+1},...,\mathcal{C}'_m}) \quad (\text{ By Def. of } d\ ) \\
= &\sum_{i\leq m}\sum_{\vec{u}} \mathbb{I}(\vec{n}', \vec{u}, a_i, t_{sa}) \times \mathcal{L}(a_i)^{\vec{h}}_{\vec{u}} \times \sum_{\mathcal{C}'_1,...,\mathcal{C}'_{i-1},\mathcal{C}'_{i+1},...,\mathcal{C}'_m} cf(\vec{u})\left(\prod_{j=1}^{m}\sum_{\{\vec{u}' : \Sigma \models \vec{h} = \vec{u} \supset [a_j]\vec{h} = \vec{u}'\}} \frac{d'(w_{\vec{u}',\mathcal{C}'_j})}{cf'(\vec{u}')}\right) \quad (\text{By Eq. (A.7)})
\end{aligned}$$

$$=\sum_{i\leq m}\sum_{\vec{u}}\mathbb{I}(\vec{n}',\vec{u},a_i,t_{sa})\times\mathcal{L}(a_i)^{\vec{h}}_{\vec{u}}\times cf(\vec{u})\frac{d'(w_{\vec{n}',\mathcal{C}'})}{cf'(\vec{n}')}\sum_{\mathcal{C}'_1,\ldots,\mathcal{C}'_{i-1},\mathcal{C}'_{i+1},\ldots,\mathcal{C}'_m}\prod_{j=1,j\neq i}^{m}\sum_{\{\vec{u}':\Sigma\models\vec{h}=\vec{u}\supset[a_j]\vec{h}=\vec{u}'\}}\frac{d'(w_{\vec{u}',\mathcal{C}'_j})}{cf'(\vec{u}')}$$

$$=\sum_{i\leq m}\sum_{\vec{u}}\mathbb{I}(\vec{n}',\vec{u},a_i,t_{sa})\times\mathcal{L}(a_i)^{\vec{h}}_{\vec{u}}\times f(\vec{u})\times\frac{d'(w_{\vec{n}',\mathcal{C}'})}{f'(\vec{n}')}\quad(\sum_{\mathcal{C}'}d'(w_{\vec{u}',\mathcal{C}'})=cf'(\vec{u}')\text{ for all }\vec{u}'\ )$$

$$=\frac{d'(w_{\vec{n}',\mathcal{C}'})}{f'(\vec{n}')}\sum_{i\leq m}\sum_{\vec{u}}\mathbb{I}(\vec{n}',\vec{u},a_i,t_{sa})\times\mathcal{L}(a_i)^{\vec{h}}_{\vec{u}}\times f(\vec{u})$$

$$=\frac{d'(w_{\vec{n}',\mathcal{C}'})}{f'(\vec{n}')}\times\left(f'(\vec{n}')-\sum_{i>m}\sum_{\vec{u}}\mathbb{I}(\vec{n}',\vec{u},a_i,t_{sa})\times\mathcal{L}(a_i)^{\vec{h}}_{\vec{u}}\times f(\vec{u})\right)\quad(\text{ By Def. of } f')$$

$$=\frac{d'(w')}{f'(\vec{n}')}\times\left(f'(\vec{n}')-\sum_{i>m}\sum_{\vec{u}}\mathbb{I}(\vec{n}',\vec{u},a_i,t_{sa})\times\mathcal{L}(a_i)^{\vec{h}}_{\vec{u}}\times f(\vec{u})\right)$$

On one hand, we have

$$P_1=\frac{d'(w')}{f'(\vec{n}')}\times\left(f'(\vec{n}')-\sum_{i>m}\sum_{\vec{u}}\mathbb{I}(\vec{n}',\vec{u},a_i,t_{sa})\times\mathcal{L}(a_i)^{\vec{h}}_{\vec{u}}\times f(\vec{u})\right)>\frac{d'(w')}{f'(\vec{n}')}\times\left(f'(\vec{n}')-\frac{\nabla\epsilon}{2cN}\right)\quad(\text{By Eq. (A.6)})$$

$$=d'(w')-\frac{\nabla\epsilon d'(w')}{2cNf'(\vec{n}')}\geq d'(w')-\frac{\nabla\epsilon}{2Nf'(\vec{n}')}\quad(\textsc{Norm}(d',\mathcal{W}_{\text{TRUE}}),c)$$

$$\geq d'(w')-\frac{\epsilon}{2N}\quad(\ \nabla\leq f'(\vec{n}')\text{ by Eq. (A.5)}\ )$$

On the other hand we have

$$P_1=\frac{d'(w')}{f'(\vec{n}')}\times\left(f'(\vec{n}')-\sum_{i>m}\sum_{\vec{u}}\mathbb{I}(\vec{n}',\vec{u},a_i,t_{sa})\times\mathcal{L}(a_i)^{\vec{h}}_{\vec{u}}\times f(\vec{u})\right)\leq\frac{d'(w')}{f'(\vec{n}')}f'(\vec{n}')=d'(w')$$

For $P_2$, we have

$$P_2=\sum_{i>m}\sum_{\vec{u}}\mathbb{I}(\vec{n}',\vec{u},a_i,t_{sa})\times\mathcal{L}(a_i)^{\vec{h}}_{\vec{u}}\sum_{\{w:w_{(a_m)}=w'\text{ and }w\models\vec{h}=\vec{u}\}}d(w)\leq\sum_{i>m}\sum_{\vec{u}}\mathbb{I}(\vec{n}',\vec{u},a_i,t_{sa})\times\mathcal{L}(a_i)^{\vec{h}}_{\vec{u}}\sum_{\{w:w\models\vec{h}=\vec{u}\}}d(w)$$

$$=\sum_{i>m}\sum_{\vec{u}}\mathbb{I}(\vec{n}',\vec{u},a_i,t_{sa})\times\mathcal{L}(a_i)^{\vec{h}}_{\vec{u}}\times cf(\vec{u})<\frac{\nabla\epsilon}{2N}\leq\frac{\epsilon}{2N}$$

The last two steps are by Eq. (A.6) and $\nabla\leq 1$ respectively. Therefore, for $w'\in\{w'_1,\ldots,w'_N\}$, with all the equities and inequalities above, we have

$$d'(w')-\frac{\epsilon}{2N}\leq d_{t_{sa}}(w')\leq d'(w')+\frac{\epsilon}{2N}\tag{A.8}$$

By Proposition 8 and $\textsc{Norm}(d',\mathcal{W}_{\text{TRUE}},c)$, we have

$$\sum_{i>N}d_{t_{sa}}(w'_i)=c-\sum_{i\leq N}d_{t_{sa}}(w'_i)\leq c-\sum_{i\leq N}(d'(w'_i)-\frac{\epsilon}{2N})\qquad(\text{Inequality (A.8)})$$

$$=c-\sum_{i\leq N}d'(w'_i)+\frac{\epsilon}{2}=\sum_{i>N}d'(w'_i)+\frac{\epsilon}{2}\qquad(\textstyle\sum_{w'}d'(w')=c)$$

Finally, we have

$$\rho(d_{t_{sa}},d')=\sum_{i=1}^{\infty}|d_{t_{sa}}(w'_i)-d'(w'_i)|\leq\sum_{i\leq N}|d_{t_{sa}}(w'_i)-d'(w'_i)|+\sum_{i>N}d_{t_{sa}}(w'_i)+\sum_{i>N}d'(w'_i)$$

$$\leq\frac{\epsilon}{2N}*N+\sum_{i>N}d_{t_{sa}}(w'_i)+\sum_{n>N}d'(w'_i)\leq\frac{\epsilon}{2N}*N+2\sum_{i>N}d'(w'_i)+\frac{\epsilon}{2}<\frac{\epsilon}{2}+\frac{\epsilon}{2}+\frac{\epsilon}{2}=\frac{3\epsilon}{2}$$

The last step is due to $\sum_{w'\in\{w'_i:i>N\}}d'(w')<\frac{\epsilon}{4}$. Since $\epsilon$ could be arbitrarily small, we have $\rho(d_{t_{sa}},d')<\epsilon$. □

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